THE ELEMENTS OF ALGEBRA.

THE ELEMENTS OF ALGEBRA

IN A

New and Easy Method;

WITH THEIR

USE and APPLICATION,

IN THE

Solution of a great Variety of Arithmetical and Geometrical Questions;

By general and universal RULES.

To which is prefixed an

INTRODUCTION,

CONTAINING

A Succinct HISTORY of this SCIENCE.

By *NATHANIEL HAMMOND*,
Of the BANK.

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MDCCLXXII.

There are two Things abſolutely neceſſary, to make the Acquiſition of any Science as eaſy as its Nature will admit. Firſt, the Diſpoſition of the Work, ſo that the Rules be clear and diſtinct; and then the Illuſtration of theſe Rules by a ſufficient Number of proper and pertinent Examples. And tho' the excellent *Elements* of the judicious EUCLID are of a different Nature, yet in this I have induſtriouſly ſtudied to imitate him, that I propoſe no new Rule or Article in this *Science of Inveſtigation*, till it become neceſſary to carry the Learner to a further Degree of Knowledge.

Science may be compared to a highly finiſhed Pile of Building, all the Parts of which being diſpoſed in the moſt exact Symmetry, they muſt affect our Perception, and gratify our internal Senſation with a more exquiſite Pleaſure, than if viewed in a ſeparate State: For in ſuch a State, to all but the Learned, they would appear broken and inconnected Materials of a mighty Structure, which the Mind wanting Power to conceive, could enjoy no Satisfaction in the Contemplation of ſuch a Train of imperfect and confuſed Ideas. But, when thus exhibited in their true Proportion, it will be eaſy, even for the youngeſt Scholar, to gain a perfect Notion of each, and, as he advances, a gradual Comprehenſion of the Beauty reſulting from their Connection, and how they mutually aſſiſt and ornament each other.

In teaching the abſtract Sciences, Examples have a ſtrong as well as natural Tendency to illuſtrate the Precepts concerning our abſtract Reaſonings, eſpecially in this Science of *Inveſtigation*: In the *Synthetic* Method, or Method by Demonſtration, one Example

Example or Propofition is fufficient, for a Number of the fame Propofitions is only a Repetition of the fame fucceffive Train of Ideas; whereas, in the *Analytic* Method, or Method by *Inveftigation*, the fame Conclufion is gained, tho' there may be a great Variety in the Connection of the Reafoning, by the different Difpofition of the Ideas, notwithftanding they are directed by the fame general Rule.

And as the principal Difficulty in this Science, is acquiring the Knowledge of folving of Queftions, I have given a great Variety of thefe in refpect to Numbers and Geometry, and their Solutions I chofe to give in the moft particular, diftinct, and plain Manner; and for which the Reader will find full and explicit Directions. As it is prepofterous and abfurd, to expect a Perfon to be a *Critic* in any Language as foon as he has paffed thro' his Grammar; fo, I cannot help thinking it wrong to expect a Learner fhould fee the Reafon of particular elegant Methods of Solution, before he has practifed a general and univerfal Method; but after the general Rules are become eafy and familiar, the Learner may then apply himfelf to the particular Methods. And I know of no Work that has illuftrated and exemplified the general and univerfal Rules in fo copious a Manner, as will be found in the following Sheets.

In the Arithmetical Operations, the Decimal Fractions are continued to two Places only, thefe being fufficient to fhew the Reader, that if the Queftion admits not of an exact Anfwer, he is yet near the Truth, and may profecute the Anfwer to any required Degree of Exactnefs. I have avoided all tedious Numerical Calculations, as they have no Tendency

Tendency to increaſe the Learner's Knowledge in *Algebra*.

The Rules of Vulgar Fractions in *Algebra* are omitted, being generally very perplexing to Learners; but as I have given ſufficient Directions how they are managed whenever they occur in the Solution of any Queſtion, the Reader will find no Difficulty in reducing an Equation with Fractional Quantities.

It is neceſſary my Reader ſhould underſtand Vulgar and Decimal Fractions in common Arithmetick, and the Extraction of the Square Root, and then I know no Reaſon why a Perſon may not make himſelf a perfect Maſter of the following Work, excepting the Geometrical Queſtions, which he may omit, and proceed to thoſe which require no Skill in Geometry; for thro' the whole, where it was neceſſary, I have given the ſame Directions, as if I was actually teaching a Scholar.

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INTRODUCTION.

AS all Arts have their Beginning, rude and weak, and reach Perfection by Degrees, ſo that, which is the Subject of the following Sheets, has been cultivated by ſo many illuſtrious Men in our own, as well as in foreign Nations, that it cannot but appear a natural Introduction to this Treatiſe, if we digeſt the Hiſtory of its Riſe and Progreſs into a ſuccinct Diſcourſe; the rather, becauſe Books of this Sort are now become very numerous in ours, as well as in other Languages, and therefore, it is the more neceſſary to record the Names of ſuch as have eminently improved ſo uſeful a Branch of Knowledge.

The Word *Algebra* is certainly derived from the *Arabic*, but there have been ſome Miſtakes as to its Meaning. When it was firſt introduced in *Europe*, it was underſtood to be the Invention of the famous Philoſopher *Geber*; and therefore *Michael Stifelius* calls it ſometimes *Regula Algebræ*, and ſometimes *Regula Gebri*, whence it is plain he underſtood by it no more than the Rule of *Geber*, or, as we uſually expreſs it, *Geber*'s Rule. But when we became better acquainted with *Arabic* Learning, this Derivation appeared ill founded: In that Language, this Art is called *Al-gjábr W'al-mokábala*, which is literally, the Art of Reſolution and Equation. Hence it is plain, we had the Word *Algebra* from the *Arabic* Name of the Art, and not from the pretended Inventor. But it may not be amiſs to obſerve, that the *Arabic* Name contains a Definition, or is rather an emphatic Decla-

ration of the Nature and End of this Science; for the *Arabic* Verb *jábara* signifies to reset, and is properly used in respect to Dislocations, and the Verb *hábala*, implies opposing, or comparing; and how applicable this is to what we call Algebra, the Reader, when he is thoroughly acquainted with this Book, will easily understand. As it became better known to the *Europeans*, it received different Names; the *Italians* stiled it *Ars magna*, in their own Language *l'Arte Magjore*, opposing to it common Arithmetick, as the lesser or minor Art. It was also called *Regula Cosæ*, the Rule of *Coss*, for an odd Reason: The *Italians* make use of the Word *Cosa*, to signify what we call the Root, and from thence, this Kind of Learning being derived to us from them, the Root, the Square, and the Cube, were called *Cossick* Numbers, and this Science the Rule of *Coss*. I should not have dwelt so long on so dry a Subject, but that it is absolutely necessary for the understanding what follows.

It is a Point still disputed, whether the Invention of Algebra ought to be ascribed to the *Oriental* Philosophers, or to the *Greeks*; but it is a Thing certain, that we received it from the *Moors*, who had it from the *Arabians*, who own themselves indebted for it to the *Persians* and *Indians*; and yet, which is strange enough, the *Persians* refer the Invention to the *Greeks*, and particularly to *Aristotle*. Yet, notwithstanding this, it must be allowed that the Algebra taught us by the *Arabians* differs very much from that contained in the Works of *Diophantus*, the eldest *Greek* Author on this Art, which is now extant, and which was discovered and published long after the Algebra taught by the *Arabians* had been studied and improved in the West. But all these Difficulties, which have given some great Men so much Trouble, may be easily surmounted, if we suppose that the Invention was originally taken from the *Greeks*, and new modelled by the *Arabians*, in the same Manner as we know that common Arithmetick was; for this, which is at least extremely probable, makes the whole plain and clear,

clear, and leaves us at liberty to purſue the Progreſs of this Art from the firſt printed Treatiſes about it.

Lucas Paciolus, a Franciſcan Friar, commonly known by the Name of *Lucas de Burgo Sancti Sepulchri*, publiſhed at *Venice*, under the Title of, *A Compleat Treatiſe of Arithmetick and Geometry*, *Proportions and Equations*, the firſt Book at preſent extant on this Subject. It was printed ſo early as 1494, and is a very correct Treatiſe. He aſcribes the Invention of Algebra to the *Arabians*, uſes their Method, and treats very clearly of Quadratic Equations. After him, ſeveral Authors wrote on the ſame Subject in *Italy*, and in *Germany*; but ſtill the Art advanced little 'till the famous *Jerom Cardan* printed, at *Nuremberg* in 1545, in Folio, a Treatiſe with this Title, *Artis magnæ*, *ſive de Regulis Algebraicis Liber unus*; and ſoon after a ſmaller Piece, with the Title of *Sermo de Plus & Minus*, wherein were contained Rules for reſolving Cubic Equations, which have ſince been called *Cardan*'s Rules, though they were not invented by him, but, as himſelf owns, by *Scipio Ferreus* of *Bononia*, and *Tartalea*. The next celebrated Writer was a *French* Monk, whoſe Name was *Baeton*, better known to the Learned by his *Latin* Appellation of *Buteo*; he publiſhed in 1559 his *Logiſtica*, in which there was a Treatiſe of Algebra which gained him great Reputation: Yet his Excellency lay in a clear and copious Manner of writing, nor does it appear that he added any thing to what had been already diſcovered, except ſome Corrections as to *Tartalea*'s Method of managing Cubic Equations.

Hitherto nothing was known in *Europe* of the *Greek* Analyſis, but in 1575 *Xilander* publiſhed *Diophantus*, or at leaſt a Part of his Works, which are ſtill remaining; and this quickly changed the Face of Things, for it preſently appeared that his was a nearer and more eaſy Method, and withal opened a Path to much greater Diſcoveries, which was the Reaſon that ſucceeding Algebraiſts quitted the Terms made uſe of by *Arabic* Writers, and followed his. The Time in which *Diophantus* flouriſhed

is not thoroughly ſettled. *Voſſius* thinks he lived in the ſecond Century, but others place him in the fourth. His Works were known to the *Arabians*, and tranſlated by them; nay, it is ſaid, that they have ſtill thoſe ſeven Books of his Arithmetick, which are loſt to us. The famous *Arabian* Hiſtorian *Abul Pharaijus*, whoſe Works were publiſhed by the learned *Pocock*, not only mentions him, but aſcribes to him the Invention of Algebra; but in this he is to be underſtood, as writing according to the Lights he had; for tho' it be true, that *Diophantus Alexandrinus* is the oldeſt Author we have which treats expreſsly of the *Analytic Art*, yet the Footſteps thereof are viſible in much older Writers. *Theo*, who is thought to have explained the five firſt Propoſitions of the thirteenth Book of *Euclid* in the Analytic Way, gives the Honour of this Invention to *Plato*; and indeed, it ſeems very agreeable to his Genius, and Method of Reaſoning on Mathematical Subjects. By the Junction of both Lights, and a proper Connection of the *Arabic* Method of Inveſtigation with the *Greek* Terms, which were ſhorter and eaſier, Algebra quickly became a much more uſeful, as well as conſiderable Science, than it was before.

In our own Country, the firſt Writer upon Algebra that we know of was Dr. *Robert Record*, a Phyſician, who diſtinguiſhed himſelf in the Reign of Queen *Mary*, by his Skill in the Mathematicks. He firſt publiſhed a Treatiſe of Arithmetick, which continued the Standard in that Branch of Knowledge for many Years, and in 1557 he ſent abroad a ſecond Part, under the Title of *Cos Ingenii, or the Whetſtone of Wit*, which is a Treatiſe of Algebra; the Word *Cos* alluding to *Coſſick* Numbers, or the Rule of *Cos*, by which Name, as we have before ſhewn, this Art was known abroad. This Treatiſe is really a great Curioſity, conſidering the Time in which it was publiſhed, and together with his other Works, muſt give us a high Idea of this Man's Induſtry and Application, whoſe Memory notwithſtanding is almoſt buried in Oblivion. But, notwithſtanding the early Publication of

of this Piece, and that ſome *Engliſh* Gentlemen had in their Travels acquired ſome Knowledge of this Kind, as appears by a *Spaniſh* Treatiſe of Algebra, publiſhed by *Pedro Nunnez* in 1567, yet it continued to be ſo little cultivated in *England*, that *John Dee*, in his Mathematical Preface prefixed to Sir *Henry Billingſley*'s Tranſlation of *Euclid*, printed at *London* in 1570, ſpeaks of it in very high Terms, and as a Myſtery ſcarce heard of by the Studious in the Mathematicks here. It is however plain, from ſome of his Annotations on *Euclid*, that he was tolerably verſed therein, and was even acquainted with the Manner of applying it to Geometry. In 1579 *Leonard Digges*, a great Mathematician for thoſe Times, printed a Treatiſe of Algebra in his *Stratioticos*; after which it came to be better known and more ſtudied, to which contributed not a little, the Improvements made by the Author I am next to mention.

Francis Viete, better known by his *Latin* Name of *Franciſcus Vieta*, was a Native of *Poitou*, in *France*, and Maſter of Requeſts to Queen *Margaret*, firſt Wife to King *Henry* IV. His Affection to the Mathematicks, and eſpecially to this Part of it, was ſo ſtrong, that he frequently paſſed three whole Days and Nights in his Study without eating, drinking, or ſleeping, except a Nod now and then upon his Elbow *. He, about the Year 1590, publiſhed a Treatiſe of Algebra in quite a new Method, and by a judicious Mixture of the *Greek* and *Arabian* Rules, with ſome Improvements of his own, introduced that Mode of Calculation which is ſtill in Uſe, under the Title of *Specious Arithmetick*. Before his Time, only unknown Quantities were marked by Letters, but ſuch as were known were ſet down in Figures according to the uſual Notation: He made uſe of Letters for both, only with this Diſtinction, that the known Quantities he repreſented by Conſonants, and the unknown by Vowels. By this Contrivance he greatly extended the Science, and which was more, ſhewed its Capacity of being farther extended. For, whereas former Algebraiſts had confined

their

* *Thuan.* Hiſt. *A. D.* 1603.

their Inveſtigations to the particular Queſtions propoſed to them, he by this Means produced Theorems capable of reſolving all Demands of a like Nature, inſtead of particular Solutions. The learned Dr. *Wallis* has accounted very clearly for the new Title which *Vieta* gave to his Algebra. The *Romans* had a Method of ſtating Law Queſtions under general Names, ſuch as *Titus* and *Sempronius*, *Caius* and *Mevius*, whence we derive our Way of uſing A, B, C, D, on ſuch Occaſions, which Method of ſtating the *Civilians* ſtile *Species*, in Oppoſition to the ſtating of *real Caſes* by *true Names*. *Vieta* having made a Change of the ſame Nature in Algebra, and being, as we obſerved before, a Lawyer by Profeſſion, he borrowed from that Science this Title of his new Invention, which was received with univerſal Applauſe. We have likewiſe many of his Works, under the Name of *Apollonius Gallus*, which he aſſumed on Account of his firſt attempting to reſtore the Works of *Apollonius Pergæus*. His Genius was ſo extenſive, and his Penetration ſo great, that it enabled him to apply his Mathematical Knowledge to moſt Subjects; of which we have a particular Inſtance, in his decyphering the Letters which paſſed between the Court of *Spain*, and the Faction of the League in *France*, notwithſtanding above five hundred different Characters were made uſe of in them. About the ſame Time flouriſhed *Raphael Bombelli*, an *Italian*, who publiſhed at *Florence* a Treatiſe of Algebra, wherein he firſt taught how to reduce a biquadratic Equation to two Quadratics, by the Help of a Cubic.

Our own Countryman, Mr. *William Oughtred*, was the next great Improver of Algebra. Building, however, on what *Vieta* had already performed. He introduced ſuch a Conciſeneſs, and withal ſo plain and perſpicuous a Method of inveſtigating Geometrical Problems, as acquired him immortal Reputation. His *Clavis Mathematicæ*, or *Key of the Mathematicks*, was firſt publiſhed in 1631, and is perhaps the cloſeſt and moſt compendious Syſtem hitherto extant.

extant. In this Work he contented himſelf with the Solution of quadratic Equations, reſerving thoſe of higher Powers for another Work, which was his *Exegeſis Numeroſa*, which in later Editions is joined to his *Clavis*. In both Pieces there were abundance of Additions and Improvements, and the Doctrine of Proportions more fully and clearly ſtated than hitherto it had been; but the greateſt Excellency in Mr. *Oughtred*'s Book, was his Application of the Analytic Method to Geometry, which he did in a Variety of Caſes, and enabled his Diſciples to proceed ſtill farther than himſelf had done. By Profeſſion he was a Clergyman, and Rector of *Albury* in *Surry*, where he gave himſelf up entirely to his Studies, and to the Converſation of a very few Friends; he lived to the Age of Fourſcore and Seven, and died then of Joy, on *May* 1, 1660, at hearing the Houſe of Commons had voted the King's Return. Some have cenſured his *Clavis* as too ſhort and obſcure, and ſo indeed it might prove for ſuch as were altogether unacquainted with theſe Studies, for whoſe Uſe it is plain enough he never deſigned it; but where Perſons are acquainted with the Elements of Geometry and Algebra, and have that Sagacity and Attention which is neceſſary to make any conſiderable Progreſs in this Sort of Learning, Mr. *Oughtred*'s Key will be ſtill found a very uſeful Book, and its Style the moſt perfect in its Kind that has ever been uſed.

Contemporary with him was Mr. *Thomas Harriot*, an excellent Mathematician, and who made ſtill greater Improvements in this Science. He is placed after *Oughtred*, tho' he died long before him, becauſe his Book was not publiſhed till ſome Time after the firſt Edition of *Oughtred*'s *Clavis*. It was then printed in a thin Folio by the Care of Mr. *Walter Warner*, under the Title of *Artis Analyticæ Praxis ad Æquationes Algebraicas novâ, expeditâ, & generali Methodo, reſolvendas, Tractatus poſthumus, &c. i. e.* A Treatiſe of the Analytic Art, containing a new, expeditious, and general Method of reſolving Equations, a poſthumous Tract, by the late learned Mr. *Thomas Har-*

riot.

riot. The Publisher, Mr. *Warner*, prefixed a Preface of his own, containing a very judicious, tho' very concise, Representation of the several Parts of Algebra, their Nature and Dependance on each other, the Extent and Usefulness of this Art, and the Progress thereof to that Time. In Mr. *Harriot*'s Book, Algebra takes a new Form, and from him alone it met with more Improvement than from all who had studied, or at least all who had written upon it, before him. He was indeed one of the greatest Men this Nation ever produced, and great Pity it was, that this Work of his did not appear in his Life-time, or that his other Pieces, which were of infinite Value, should be buried in Oblivion. The true Cause of the former seems to have been his Course of Life; he was a Dependant on the Earl of *Northumberland* and Sir *Walter Raleigh*, and afterwards upon Sir *Thomas Aylesbury*, to whom, if I am rightly informed, he left many of his Writings, and, as I hinted, the Reason of his not publishing them in his Life-time, seems to have been his Deference for his Benefactors. Happy had it been, if the rest of the Mathematical Works he left had been sent abroad (as in his Preface he seemed to promise they should) by the intelligent Editor of this excellent Work.

It is divided into two Parts; and the Author begins his Improvements by removing every thing that was useless, superfluous, or inelegant in former Methods; thus instead of Capitals, he introduced small Letters; instead of the Terms, Squares, Cubes, Sursolids, *&c.* and their Contractions, he brought in the Powers themselves, which made the Operations much more easy, natural, and perspicuous than they were before. Having thus established a plain and accurate Notation, he proceeds to a Multitude of new Discoveries, of which, to the Number of twenty-three, the Reader may find a full, distinct, and very judicious Account, in the celebrated Treatise of Dr. *Wallis*. From this admirable Piece of Mr. *Harriot*'s, *Des Cartes* took all the Improvements he pretended to

make, as the Doctor justly observes, and of which I shall furnish the Reader with some concise, and I think conclusive Proofs. *First*, It appears from all the Accounts we have of the Life of *Des Cartes*, that he was here in *England* when *Harriot*'s Book was published, which being written in *Latin*, in a Branch of Learning about which that great Man was then very sedulous, it is easy to conceive that he was one of its most early Perusers; *Secondly*, It is certain that he did not publish any thing on this Subject before that Year; *Thirdly*, His Treatise of Geometry, wherein these new Improvements first appeared, was printed in *French* in 1637 without his Name, which in all Probability was to try what Opinion the World would have of them, and whether any of the *French* Mathematicians could discern whence they were taken; *Fourthly*, Though he suffered the two first Parts of his Book to be published in *Latin*, with his Name, in 1644; yet the third Part, relating to Geometry, did not appear till 1649, when it was published by *Francis Van Schooten*. These are probable Reasons only, but then, *Fifthly*, He follows *Harriot* distinctly in *Nineteen* several Discoveries; which that they should be made in the same Method and Manner, (except a few Mistakes) without consulting Mr. *Harriot*, is altogether incredible, and was so held to be even by his own Countrymen, when, thro' the Information of the Honourable Mr. *Cavendish*, they were made acquainted with Mr. *Harriot*'s Book; *Sixthly*, There are some little Changes, particularly in the Marks made use of by *Des Cartes*, and which were never followed by any body, that plainly intimate he only introduced them, in order to disguise his Method; *Seventhly*, It appears that *Des Cartes* himself was acquainted with the Charge brought against him upon this Head, and yet he never thought fit to justify himself, nor did ever so much as declare that he had not seen the Book he was said to have copied. On the whole therefore, there is all the Reason in the World to believe, that the Honour due to the great Improvement of this Science,

which fitted it for all that it has received ſince, from Foreigners or *Engliſhmen*, belongs to our Author *Harriot*, and not to *Des Cartes*, who only accommodated theſe Diſcoveries to Geometrical Subjects.

After him Dr. *John Pell*, who was Reſident for the *Commonwealth* of *England* in *Switzerland*, publiſhed ſome new Diſcoveries. The Method he took of doing it was this, he recommended to Mr. *Thomas Brancker* a Treatiſe of Algebra written in the *German* Language by *Rhonius*, which when he had tranſlated, the Doctor reviſed, altered and added to it. In this Piece there are a great many curious Things relating eſpecially to *Diophantine Algebra*, but delivered very obſcurely, inſomuch, that the learned Dr. *Wallis* ſeems to be in doubt, whether himſelf had reached Dr. *Pell*'s true Meaning. Yet, to this Gentleman, who wrote in ſo perplexed a Way, we ſtand indebted for the Invention of the Regiſter; a Method of great Uſe, eſpecially to Beginners, the Practice of which was what chiefly recommended *Kerſey*'s Algebra, and which is conſtantly and judiciouſly preſerved throughout the following Treatiſe. It is very likely, that the Darkneſs complained of in Dr. *Pell*'s Writings might be owing to his Circumſtances as well as Temper, for he was a very bad Œconomiſt, not through any Vice or Extravagancy, but by a Neglect of his private Affairs, and ſpending all his Time in Study.

As for the Rules of *John Van Hudde*, Mr. *Merry*, *Eraſmus Bartholine*, Mr. *Hugens*, and others, I do not take Notice of them, becauſe in reality they are no more than Improvements on, or Deductions from, *Harriot*. The ſame Thing may be ſaid of what has been written by Meſſ. *Farmat*, *de Billy*, *Fernicle*, and other *French* Mathematicians, who only propoſed Problems for other People to reſolve, and reſerved their own Methods of Solution as impenetrable Secrets: A Practice, which, however it might intitle them to the Admiration of the Age in which they lived, can give them no juſt Claim to the Praiſe of Poſterity; ſince if we reap any Benefit from their Diſcoveries,

veries, it is indirectly, and in a Manner againſt their Intentions.

Dr. *Wallis* himſelf has alſo made ſome very conſiderable Improvements in this Science, eſpecially in reſpect to impoſſible Roots in ſuperior Equations; and what he left unperfected has been ſupplied by the ingenious Mr. *Abraham De Moivre*, whoſe accurate Performance on that Subject has been lately publiſhed, in the Algebra of Dr. *Saunderſon*.

In 1655 Dr. *Wallis* publiſhed his *Arithmetica Infinitorum*, in which he ſquared a Series of Curves, and ſhewed that if this Series could be interpolated in the middle Spaces, the Interpolation would give the Quadrature of the Circle. This Treatiſe fell into the Hands of the ingenious Sir *Iſaac* then Mr. *Newton*, in the Year 1664, when that Gentleman was about Two and Twenty; and he by a Sagacity peculiar to himſelf, and which can never be enough admired, derived from this Hint his celebrated Method of Infinite or Converging Series. In 1665, he computed the Area of the Hyperbola by this Series to Fifty-two Figures, which having communicated to Dr. *Barrow*, he prevented Mr. *Nicholas Mercator*'s running away with the Reputation of this Diſcovery, who in 1668 publiſhed the Quadrature of the Hyperbola by an infinite Series. This was received with univerſal Applauſe, and yet Mr. *Newton* far exceeded him; ſince, without ſtopping at the Hyperbola, he extended this Method by general Forms to all Sorts of Curves, even ſuch as are Mechanical, to their Quadratures, Rectifications, and Centers of Gravity, to the Solids formed by their Rotations, and to the Superficies of thoſe Solids; ſo that ſuppoſing their Determinations to be poſſible, this Series ſtopped at a certain Point, or at leaſt their Sums were given by ſtated Rules. But if the abſolute Determinations were impoſſible, they could yet be infinitely approximated, as he likewiſe ſhewed, and which, as a *French* Writer juſtly obſerves, is the happieſt and moſt refined Contrivance for ſupplying the Defects of human Knowledge, that Man's Imagination could poſſibly invent. It is alſo certain, that he attained his

Invention of Fluxions by that Time he was Four and Twenty, but his Modesty was so great, that he forbore to publish his Discovery, which was the sole Reason that the Honour of it was ever disputed with him.

In 1707, he first published a System of Algebra under the Title of *Universal Arithmetick*, and in 1722 gave another Edition of it, wherein are contained all his Improvements in that Art.

From the Rules by him laid down, still farther Lights were struck out by succeeding Mathematicians, such as Dr. *Edmund Halley*, who published in the *Philosophical Transactions*, a Method of finding the Roots of Equations without any previous Reduction, and the Construction of Equations of the 3d and 4th Power, by the Help of a Circle and Parabola. Mr. *J. Colson*, who obliged the learned World with a universal Resolution, Geometrical and Mechanical, of Cubic and Biquadratic Equations. Mr. *Colin Mac Laurin*, in his Treatise of impossible Roots, and many others too long to be enumerated here.

But after all these Discoveries and Improvements, there has still been a general Complaint, that hitherto we have had no Book of Algebra plain enough to instruct such as are inclined to study this Science without farther Assistance, or who live in Places where it is not to be had. To obviate this Objection, the following Treatise was drawn up, which will be found to contain a clear and copious System of Algebra, delivered in so easy and natural a Method, and with such Perspicuity and Condescension to the Feebleness of the Understanding, when first applied to this kind of Study, that I felicitate myself on having prevailed upon its Author to make it publick, as I am persuaded it will be of general Use, in preventing young People from being discouraged at their first Entrance into Algebra, which has hitherto hindered Numbers from cultivating their Inclinations to the Mathematicks.

THE

CONTENTS.

Division,

ALGEBRA.

A L G E B R A.

AVING given the Reader an Hiſtorical Account of this Science in the Introduction, we are now to explain the Signs and Characters uſed by Analytic Writers, and mention thoſe Axioms or Self-evident Principles of Truth and Certainty, which are the Foundations of this celebrated Science.

Signs.	*Names.*	*Significations.*
$+$	*Plus or more.*	The Sign of *Addition*; as $8+4$, is 8 is to be added to 4, and $m+n$ ſignifies the Number repreſented by m, is to be added to the Number repreſented by n; again, $2+3+5$, ſignifies they are all to be added into one Sum, and $b+m+d$ ſignifies that the Numbers repreſented by b, m, and d are to be added into one Sum.
$-$	*Minus or leſs.*	The Sign of *Subſtraction*; as $5-2$, is 5 leſs by 2, or 2 is to be ſubſtracted from 5, and $a-b$ is a leſs b, or the Number repreſented by b is to be ſubſtracted from the Number repreſented by a; and $9-2-3$, is that from 9 there is to be ſubſtracted 2, and from the Remainder 3 is to be ſubſtracted.
$\times$	*Into or with.*	The Sign of *Multiplication*; as 5×7, is 5 is to be multiplied by 7, and $a\times b$, is the Number repreſented by a, is to be multiplied by the Number repreſented by b; and $7\times 3\times 2$, is that 7, 3, and 2, are to be multiplied together, which Product is 42.

$\div$	*By.*	The Sign of *Division*; as $8 \div 4$, that is 8 is to be divided by 4, and $x \div y$, that is, the Number repreſented by x, is to be divided by the Number repreſented by y; or ſometimes they are placed like Vulgar Fractions thus $\frac{8}{4}$, that is, 8 is to be divided by 4, and $\frac{x}{y}$, that is, the Number repreſented by x, is to be divided by the Number repreſented by y.
$=$	*Equal.*	The Sign of *Equality* or *Equation*; thus $9 = 9$, that is, 9 is equal to 9, and $2 + 3 = 5$, that is, 2 added to 3, is equal to 5: Again, $m = n + y$, that is, the Number repreſented by m is equal to the Number repreſented by n, added to the Number repreſented by y; and $y - x = a + b$, that is, the Number repreſented by y being leſſened by the Number repreſented by x, the Remainder is equal to the Number repreſented by a, added to the Number repreſented by b.
$::$	*So is.*	The Sign of *Proportion*, or what is commonly called the Rule of Three, and is placed between the two middle Number thus, $3 : 5 :: 6 : 10$, that is, as 3 is to 5, ſo is 6 to 10; and $a : b :: c : d$, that is, as the Number repreſented by a is to the Number repreſented by b, ſo is the Number repreſented by c to the Number repreſented by d.
⊖	*Involution.*	The *Sign* of *Involution*, or raiſing any Number or Quantity to the *Square*, *Cube*, or any other Power; and the Heighth of the Involution is generally expreſſed by the Number after the Sign thus, 7 ⊖ 2, is 7 is to be involved to the Square or ſecond Power; and 7 ⊖ 3, is 7 is to be involved or raiſed to the Cube or third Power; and a ⊖ 2, is a is to be involved to the Square or ſecond Power.

The

ɯ — *Evolution.*

The Sign of *Evolution*, or the extracting of Roots; and the Root that is taken is likewiſe expreſſed by the Figure that follows the Sign, thus 9 ɯ 2, is the Square Root of 9 is to be extracted, and 27 ɯ 3, is the Cube Root of 27 is to be extracted, and *a a* ɯ 2, is the Square Root of *a a* is to be extracted.

$\surd$ — *Irrationality, or a Surd Root.*

The Sign of *Irrationality*, or of a Surd Root; that is, the Number or Quantity has not ſuch a Root as is required to be extracted; thus the Square Root of 2 will be expreſſed thus $\surd 2$, and the Square Root of 5 thus $\surd 5$, and the Cube Root of 4 thus $\sqrt[3]{4}$, the little Figure ſtanding over the Sign being 3, ſhews it to be the Cube Root; again, $\sqrt[3]{15}$ is the Cube Root of 15, and where there is no ſuch Figure over the Sign, it ſignifies the Square Root only.

Now before we go farther, it will be neceſſary to inform the Reader, that where any Number is joined to a Quantity, it ſhews how many Times that Quantity is taken; thus, $4a$ is *four times a*, or the Number repreſented by a is to be taken *four times*; and $7m$ is *ſeven times m*, and if y was to be taken *ſeven times*, it may be expreſſed thus $7y$.

Theſe Numbers are called *Co-efficients*, or *Fellow-Factors*, as they multiply the Quantity; and if any Quantity is without a Co-efficient, then it is always implied that *Unity*, or 1, is the *Co-efficient* of that Quantity; thus a is the ſame as $1a$, and y the ſame as $1y$; for when the Co-efficient is only *Unity*, or 1, it is generally omitted.

Quantities that are expreſſed or repreſented by ſingle Letters, or ſeveral joined together like a Word, as a, b, ab, anz, $7yz$, are called ſimple or ſingle Quantities.

But when theſe are connected by the Signs + or —, as $a+b$, $am-d$, $dn+az$, they are called compound Quantities.

And ſometimes Quantities are ſet down in the Manner of Vulgar Fractions, thus, $\frac{a}{b}$, $\frac{a+b}{n}$, $\frac{m}{x+y}$.

The

The Sign that connects the Quantities belongs to that which follows the Sign, thus, $a + b$, where the Sign $+$ belongs to the Quantity b; again, $a - c + d$, the Sign $-$ belongs to the Quantity c, and the Sign $+$ to the Quantity d.

As to those single Quantities which have no Sign before them, it is always understood they have the Sign $+$; thus a is the same as $+ a$, and m is the same as $+ m$; and therefore if single Quantities are to have the Sign $+$, it is commonly omitted, as they are usually set down without any Sign; but the Sign $-$ is never omitted, but always placed before the Quantity to which it belongs.

And in Compound Quantities, if the first or leading Quantity has no Sign, then it is always understood to have the Sign $+$, thus, $a + b$ is the same as $+ a + b$, and $a - b$ is the same as $+ a - b$; therefore in Compound Quantities, if the first or leading Quantity is to have the Sign $+$, it is generally omitted; but in these Compound Quantities, as well as in Simple Quantities, the Sign $-$ is never omitted, but always placed before the Quantity to which it belongs.

Letters set or joined together like a Word signifies the Product or Rectangle of these Letters, thus, $a\,b$ is the Product of a multiplied by b, and $d\,n\,y$ is the Product of d, n, and y, multiplied together.

The Operations in *Algebra* are founded on these *Axioms*.

AXIOM 1.

If equal Quantities are added to equal Quantities, the Sum of these Quantities will be equal.

AXIOM 2.

If equal Quantities are taken or subtracted from equal Quantities, the Quantities remaining will be equal.

AXIOM 3.

If equal Quantities are multiplied by equal Quantities, their Products will be equal.

AXIOM

AXIOM 4.

If equal Quantities are divided by equal Quantities, their Quotients will be equal.

AXIOM 5.

If there are ſeveral Quantities that are equal to one and the ſame Thing, thoſe Quantities are equal one to another.

The Reader having premiſed theſe Things, and underſtanding what the Signs are intended to expreſs, he may proceed to the Rules of the Science; and if at firſt he meets with ſome little Difficulties about the Signs and Co-efficients, I would recommend him to read the foregoing Pages again; and if that and another Eſſay or two does not remove the Difficulties of any particular Example, then to omit that and proceed to the next, in which perhaps he may ſucceed, and that may cauſe the Difficulty in the other to vaniſh.

ADDITION,

In which there are three Caſes.

(1.) *Caſe* 1. WHEN the Quantities are alike, and their Signs are both affirmative, or both negative, add the Co-efficients or prefixt Numbers together, and to their Sum join the Quantities, prefixing to them the Sign they have in the *Example*.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
To	$2a$	$5m$	$-4y$	$-2z$
Add	$3a$	$2m$	$-3y$	$-6z$
Sum	$5a$	$7m$	$-7y$	$-8z$

Exam. 1. The Co-efficients are 2 and 3, which added together make 5, to which joining *a* the Quantity, it is $5a$, and no Sign

Sign being prefixt to either $2a$ or $3a$, the *affirmative* Sign is underſtood as prefixt to both; hence $5a$, or $+5a$ is the Sum required.

Exam. 2. The Co-efficients are 5 and 2, which being added make 7, to which joining m, it is $7m$, the Sum required; for the Signs of $5m$ and $2m$ are both *affirmative*, by what was ſaid in the laſt Example.

Exam. 3. The Co-efficients are 4 and 3, which being added make 7, to which joining y it becomes $7y$; but as $4y$ and $3y$ have both the Sign — before them, therefore prefix the Sign — to $7y$, and then $-7y$ is the Sum required.

Exam. 4. The Co-efficients are 2 and 6, which added make 8, to which joining z, it becomes $8z$, and prefixing the Sign — for the Reaſon in the laſt Example, we have $-8z$, the Sum required.

	Exam. 5.	*Exam.* 6.	*Exam.* 7.	*Exam.* 8.
To	$15my$	$-14azx$	$4ady$	$-16ymd$
Add	$7my$	$-2azx$	$3ady$	$-12ymd$
Sum	$22my$	$-16azx$	$7ady$	$-28ymd$

Exam. 5. The Sum of the Co-efficients 15 and 7 is 22, to which joining my, it is $22my$, the Sum required; for $15my$ and $7my$ have both the affirmative Sign, there being no Sign prefixt.

Exam. 6. The Sum of the Co-efficients 14 and 2 is 16, to which joining azx, it is $16azx$, to which prefixing the Sign —, as both the Quantities to be added have that Sign, then is $-16azx$ the Sum required.

Exam. 7. The Sum of the Co-efficients 4 and 3 is 7, to which joining ady, it is $7ady$, and both the Quantities having the *affirmative* Sign, therefore $7ady$ is the Sum required.

Exam. 8. The Sum of the Co-efficients 16 and 12 is 28, to which joining ymd, it is $28ymd$, to which prefixing the Sign —, as both the Quantities to be added have that Sign, then is $-28ymd$ the Sum required.

	Exam. 9.	*Exam.* 10.	*Exam.* 11.	*Exam.* 12.
To	$2my$	$1an$	$21dy$	$-da$
Add	$3my$	$2an$	dy	$-da$
Sum	$5my$	$3an$	$22dy$	$-2da$

Exam.

Exam. 11. The Co-efficients are 21 and 1, for there being no Co-efficient prefixt to dy, *Unity*, or 1, is always underſtood in ſuch Caſes to be the Co-efficient; hence the Sum is $22\,dy$.

Exam. 12. There being no Co-efficient prefixt to either of the Quantities, *Unity*, or 1, is the Co-efficient to each; and 1 being added to 1 makes 2, to which joining $d\,a$, it is $2\,d\,a$, to which prefixing the *negative* Sign, we have $-\,2\,d\,a$, the Sum required.

(2.) If there are two or more Quantities connected by the Signs $+$ or $-$, and are alike to two or more Quantities connected by the Signs $+$ or $-$, they are added as in the former Examples, only taking due Care that the Quantities which compoſe their Sum are connected with their proper Signs, according to the Rule, as in the following Examples.

	Exam. 13.	*Exam.* 14.	*Exam.* 15.
To	$2\,a + 7\,b$	$6\,ma + 5\,y$	$21\,ma + 2\,yd$
Add	$3\,a + 2\,b$	$2\,ma + 3\,y$	$3\,ma + 3\,yd$
Sum	$5\,a + 9\,b$	$8\,ma + 8\,y$	$24\,ma + 5\,yd$

Exam. 13. Is $2\,a + 7\,b$ to be added to $3\,a + 2\,b$. The Quantities being diſpoſed as in the Example, it follows from former Examples that $2\,a$ being added to $3\,a$ makes $5\,a$, and $7\,b$ added to $2\,b$ makes $9\,b$; but as $7\,b$ and $2\,b$ have both the affirmative Sign, to $5\,a$ connect $9\,b$ with the Sign $+$; hence $5\,a + 9\,b$ is the Sum required.

Exam. 14. Is $6\,ma + 5\,y$ to be added to $2\,ma + 3\,y$. Now by the former Examples $6\,ma$ being added to $2\,ma$ is $8\,ma$, and $5\,y$ being added to $3\,y$ is $8\,y$; but as $5\,y$ and $3\,y$ have both the *affirmative* Sign, to $8\,ma$ connect $8\,y$ with the Sign $+$; ſo will $8\,ma + 8\,y$ be the Sum required.

Exam. 15. Is $21\,ma + 2\,yd$ to be added to $3\,ma + 3\,yd$. Now by the former Examples $21\,ma$ being added to $3\,ma$, the Sum is $24\,ma$; and $2\,yd$ being added to $3\,yd$, the Sum is $5\,yd$. But as $2\,yd$ and $3\,yd$ have both the affirmative Sign, therefore connecting $24\,ma$ and $5\,yd$ with the Sign $+$, we have $24\,ma + 5\,yd$, the Sum required.

	Exam. 16.	*Exam.* 17.	*Exam.* 18.
To	$-\,7\,da - 15\,m$	$9\,ma - 14\,nd$	$-\,2\,mn + 15\,yd$
Add	$-\,2\,da - 4\,m$	$3\,ma - 3\,nd$	$-\,4\,mn + 4\,yd$
Sum	$-\,9\,da - 19\,m$	$12\,ma - 17\,nd$	$-\,6\,mn + 19\,yd$

Exam.

Exam. 16. Is $-7\,da - 15\,m$ to be added to $-2\,da - 4\,m$. Now $7\,da$ added to $2\,da$ is $9\,da$; but as both theſe Quantities have the Sign $-$, prefix the *negative* Sign to $9\,da$, and then it is $-9\,da$. Again, $15\,m$ added to $4\,m$ is $19\,m$; and both theſe Quantities having likewiſe the *negative* Sign, prefix it to $19\,m$; whence the Sum required is $-9\,da - 19\,m$.

Exam. 17. Is $9\,ma - 14\,nd$ to be added to $3\,ma - 3\,nd$. Now $9\,ma$ added to $3\,ma$, is $12\,ma$; and both theſe Quantities having the Sign $+$, place down $12\,ma$ as in the Example: Then $14\,nd$ added to $3\,nd$, is $17\,nd$; but both theſe Quantities having the Sign $-$, place the Sign $-$ before $17\,nd$, and the Sum required is $12\,ma - 17\,nd$.

Exam. 18. Is $-2\,mn + 15\,yd$ to be added to $-4\,mn + 4\,yd$. Now $2\,mn$ added to $4\,mn$, is $6\,mn$; but both theſe Quantities having the negative Sign, prefix the Sign $-$ to $6\,mn$, and then it is $-6\,mn$. And $15\,yd$ added to $4\,yd$, is $19\,yd$; and both theſe Quantities having the affirmative Sign, prefix the Sign $+$ to $19\,yd$; hence the Sum is $-6\,mn + 19\,yd$.

	Exam. 19.	*Exam.* 20.	*Exam.* 21.
To	$9yd - 7a$	$14yd + 15a$	$-14y + d$
Add	$2yd - a$	$2yd + a$	$-y + d$
Sum	$11yd - 8a$	$16yd + 16a$	$-15y + 2d$

Exam. 19. When you come to add $-7\,a$ to $-a$, there being no Co-efficient prefixt to a, Unity, or 1, is always in ſuch Caſes the Co-efficient; and then by what has been already taught, $-7\,a$ being added to $-a$, the Sum is $-8\,a$, as in the Example.

Exam. 20. And when $15\,a$, is to be added to a, the Sum is for the ſame Reaſon $16\,a$.

Exam. 21. And $-14\,y$ being added to $-y$, the Sum is $-15\,y$, and d being added to d, for the ſame Reaſon the Sum is $2\,d$, or $+2\,d$.

(3.) *Caſe* 2. When the Quantities are alike, but the Signs are one *affirmative*, and the other *negative*, ſubſtract the leſſer Co-efficient from the greater, to the Remainder join the Quantity, and prefix to it the Sign of the greateſt Co efficient.

It is of no Signification whether the Quantity that has the greateſt Co-efficient ſtands above or below.

2

Exam.

	Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
To	$5a$	$16m$	$21ad$	$14mz$
Add	$-2a$	$-12m$	$-7ad$	$-5mz$
Sum	$3a$	$4m$	$14ad$	$9mz$

Exam. 1. The Co-efficient 2 ſubſtracted from 5 leaves 3, to which joining a it is $3a$, but the Sign of 5 the greateſt Co-efficient is *affirmative*, therefore $3a$ or $+3a$ is the Sum required.

Exam. 2. The Co-efficient 12 ſubſtracted from 16 leaves 4, to which joining m it is $4m$, but the Sign of 16 the greateſt Co-efficient is *affirmative*, therefore $4m$ or $+4m$ is the Sum required.

Exam. 3. The Co-efficient 7 ſubſtracted from 21 leaves 14, to which joining ad it is $14ad$, but the Sign of 21 the greateſt Co-efficient is *affirmative*, hence $14ad$ or $+14ad$ is the Sum required.

Exam. 4. The Co-efficient 5 ſubſtracted from 14 leaves 9, to which joining mz it is $9mz$, but the Sign of 14 the greateſt Co-efficient is *affirmative*, hence $9mz$ or $+9mz$ is the Sum required.

	Exam. 5.	Exam. 6.	Exam. 7.	Exam. 8.
To	$-14m$	$-9y$	$5z$	$9am$
Add	$7m$	$2y$	$-9z$	$-14am$
Sum	$-7m$	$-7y$	$-4z$	$-5am$

Exam. 5. The Co-efficient 7 ſubſtracted from 14 leaves 7, to which joining m it is $7m$, but the Sign of 14 the greateſt Co-efficient being —, prefix that Sign to $7m$, then is $-7m$ the Sum required.

Exam. 6. The Co-efficient 2 ſubſtracted from 9, there remains 7, to which joining y it is $7y$, but the Sign of 9 the greateſt Co-efficient being —, prefix that Sign to $7y$, and we have $-7y$, the Sum required.

Exam. 7. The Co-efficient 5 ſubſtracted from 9 leaves 4, to which joining z it is $4z$, but the Sign of 9 the greateſt Co-efficient being negative, prefix the Sign — to $4z$, and we have $-4z$, the Sum required.

Exam. 8. The Co-efficient 9 ſubſtracted from 14 leaves 5, to which joining am it is $5\,am$, but the Sign of 14 the greateſt Co-efficient being negative, prefix the Sign — to $5\,am$, and we have $-5\,am$, the Sum required.

	Exam. 9.	*Exam.* 10.	*Exam.* 11.	*Exam.* 12.
To	$7\,am$	$-8\,ad$	$-14\,ym$	$-ay$
Add	$-am$	$9\,ad$	$16\,ym$	$7\,ay$
Sum	$6\,am$	ad	$2\,ym$	$6\,ay$

Exam. 9. The Co-efficient of $-am$ being Unity, or 1, which ſubſtracted from 7 leaves 6, to which joining am it is $6\,am$, prefixing to it the Sign of 7, the greateſt Co-efficient, we have $6\,am$ or $+6\,am$ the Sum required.

Exam. 10. The Co-efficient 8 ſubſtracted from 9 leaves 1, to which joining ad we have $1\,ad$ or ad, which having already the Sign of 9, the greateſt Co-efficient, hence ad is the Sum required.

Exam. 12. The Co-efficient of $-ay$ being Unity, or 1, which ſubſtracted from 7 leaves 6, to which joining ay it is $6\,ay$, which having the ſame Sign with 7, the greateſt Co-efficient, $6\,ay$ is the Sum required.

4. And if there are ſeveral Quantities connected by the different Signs of + and —, to be added to ſeveral Quantities connected by the different Signs of + and —, the Quantities being alike, are added as in the ſecond Article, only taking Care to prefix the Signs, according to the Directions in the firſt and third Articles.

	Exam. 13.	*Exam.* 14.	*Exam.* 15.
To	$14\,a+7\,m$	$-15\,my-14\,az$	$17\,ay+8\,am$
Add	$-8\,a-3\,m$	$7\,my+12\,az$	$-3\,ay-5\,am$
Sum	$6\,a+4\,m$	$-8\,my-2\,az$	$14\,ay+3\,am$

Exam. 13. Is $14\,a+7\,m$ to be added to $-8\,a-3\,m$. Now by the Rule at Art. 3. the Difference between the Co-efficients 14 and 8 is 6, to which joining a it is $6\,a$, but 14 the greateſt Co-efficient having the *affirmative* Sign, hence $6\,a$ is the Sum of $14\,a$ added to $-8\,a$. And the Difference between 7 and 3 the Co-efficients of m being 4, to which joining m it is $4\,m$, but as 7 the greateſt Co-efficient has the *affirmative* Sign, therefore to $6\,a$ connect $4\,m$ with the Sign +, ſo is $6\,a+4\,m$ the Sum required.

Exam.

Exam. 14. Where $-15my-14az$ is to be added to $7my+12az$. Now the Difference between 15 and 7 the two Co-efficients of my is 8, to which joining my it is $8my$, but as 15 the greateſt Co-efficient hath the *negative* Sign, therefore prefix the Sign — to $8my$, and it is $-8my$: And the Difference between 14 and 12 the two Co-efficients of az being 2, to which joining az it is $2az$, but as 14 the greateſt Co-efficient has the *negative* Sign, therefore to $-8my$ connect $2az$ with the Sign —, ſo is $-8my-2az$ the Sum required.

Exam. 15. The Difference between 17 and 3 the two Co-efficients of ay is 14, to which joining ay it is $14ay$, but as 17 the greateſt Co-efficient has the *affirmative* Sign, therefore place down $14ay$ or $+14ay$. And the Difference between 8 and 5 the two Co-efficients of am is 3, to which joining am it is $3am$, but as 8 the greateſt Co-efficient has the *affirmative* Sign, therefore prefix the Sign + to $3am$, ſo is $14ay+3am$ the Sum required.

	Exam. 16.	*Exam.* 17.	*Exam.* 18.
To	$-7a+16m$	$-15y+7p$	$7am-16y$
Add	$3a-4m$	$7y-11p$	$-11am+18y$
Sum	$-4a+12m$	$-8y-4p$	$-4am+2y$

Exam. 16. By Art. 3. the Difference between 7 and 3 the two Co-efficients of a is 4, to which joining a it is $4a$, but as 7 the greateſt Co-efficient has the *negative* Sign, therefore prefix the Sign — to $4a$, and it is $-4a$. And the Difference between 16 and 4 the two Co-efficients of m is 12, to which joining m it is $12m$, but 16 the greateſt Co-efficient having the *affirmative* Sign, prefix the Sign + to $12m$, ſo is $-4a+12m$ the Sum required.

Exam. 17. By Art. 3. the Difference between 15 and 7 is 8, to which joining y it is $8y$, but 15 the greateſt Co-efficient having the *negative* Sign, prefix the Sign — to $8y$, and it is $-8y$. And the Difference between 7 and 11 the two Co-efficients of p is 4, to which joining p it is $4p$, but as 11 the greateſt Co-efficient has the *negative* Sign, therefore prefix the Sign — to $4p$, and it is $-4p$, ſo is $-8y-4p$ the Sum required.

Exam. 18. By Art. 3. the Difference between 7 and 11 is 4, to which joining am it is $4am$, but as 11 the greateſt Co-efficient has the *negative* Sign, therefore prefix the Sign — to $4am$, and it is $-4am$. And the Difference between 16

 and

and 18 is 2, to which joining y it is $2y$, but as 18 the greateſt Co-efficient has the *affirmative* Sign, therefore prefix the Sign $+$ to $2y$, ſo is $-4am+2y$ the Sum required.

	Exam. 19.	*Exam.* 20.	*Exam.* 21.
To	$14my-ma$	$-5yd+15z$	$-14dy+5mp$
Add	$-3my+4ma$	$yd-3z$	$dy-mp$
Sum	$11my+3ma$	$-4yd+12z$	$-13dy+4mp$

Exam. 19. The Co-efficient of $-ma$ is 1, which being by Art. 3. ſubſtracted from 4 leaves 3, to which joining ma it is $3ma$, as in the Anſwer, and by the ſame Method in

Exam. 20. If $-5yd$ is added to yd or $1yd$, the Sum is $-4yd$; and likewiſe in

Exam. 21. If $-14dy$ is added to dy or $1dy$, the Sum is $-13dy$.

5. If the Quantities are alike and the Co-efficients are equal, but the Signs are one *affirmative*, and the other *negative*, theſe being added together deſtroy each other, or the Sum of them is a *Cypher* or *nothing*.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4
To	$7a$	$-5y$	$14m$	$5ya$
Add	$-7a$	$5y$	$-14m$	$-5ya$
Sum	0	0	0	0

Exam. 1. By Art. 3. the Signs being unlike the Co efficients are to be ſubſtracted, but 7 taken from 7 leaves 0, and if to this we join a it is $0a$, or *no times a*, that is, the Quantity a is to be taken no times or not at all, which is the ſame as *nothing*: So in the fourth Example, if 5 is ſubſtracted from 5, there remains 0, or *nothing*, to which if we join ya, we then have *no times y a*, or *nothing*.

(6.) *Caſe* 3. When the Quantities are unlike, that is, the Letters are different, then ſet them down one after the other, with the ſame Co-efficients and Signs they have in the Example, and this is the Sum required.

And they may be ſet in any Order, that is, any Quantity may be ſet firſt, in the middle or laſt, it being not material how they are ranged, ſo as they are but connected with their proper Signs.

Exam.

	Exam. 1.	Exam. 2.	Exam. 3.
To	$2a$	$3m$	$a+d$
Add	$3d$	$5a$	$2y$
Sum	$2a+3d$	$3m+5a$	$a+d+2y$

Exam. 1. The Quantities or Letters being unlike, I place down $2a$, and becauſe $3d$ has the Sign $+$, therefore after the $2a$ put $+3d$, ſo is $2a+3d$ the Sum required.

Exam. 2. Having put down the $3m$, after that put $+5a$ the other Quantity with its Sign, ſo is $3m+5a$ the Sum required.

Exam. 3. Having put down a, after that put $+d$, and after that $+2y$, ſo is $a+d+2y$ the Sum required.

	Exam. 4.	Exam. 5.
To	$2a-7m$	$2a+15$
Add	$3y+5z$	$z-7d$
Sum	$2a-7m+3y+5z$	$2a+15+z-7d$

Exam. 4. Begin and place down $2a$, after that $-7m$, after that $+3y$, and after that $+5z$, ſo is $2a-7m+3y+5z$ the Sum required.

Exam. 5. Begin and place down $2a$, after that $+15$, after that $+z$, and after that $-7d$, ſo is $2a+15+z-7d$ the Sum required.

To	$7m+15y$	$-15m+7a$
Add	$-4a+mn$	$8y-2b$
Sum	$7m+15y-4a+mn$	$-15m+7a+8y-2b$

To	$16+7m$	$-14m-15y$
Add	$-2a-8d$	$a-7$
Sum	$16+7m-2a-8d$	$-14m-15y+a-7$

Examples wherein all the foregoing Caſes are promiſcuouſly uſed.

	Exam. 1.	Exam. 2.
To	$7a-15d+m$	$-8a+7m-21x$
Add	$5a+18d$	$11a-12m+5y$
Sum	$12a+3d+m$	$3a-5m-21x+5y$

Exam. 1. $7a$ added to $5a$ makes $12a$, by Art. 1. and $-15d$ added to $18d$ makes $3d$, by Art. 3. and there being no Quantity like m, that muſt be placed by itſelf, by Art. 6. and connecting

ing these Quantities with their proper Signs we have $12a + 3d + m$, the Sum required.

Exam. 2. $-8a$ added to $11a$ makes $3a$, by Art. 3. and $7m$ added to $-12m$ is $-5m$, by the same, but $21x$ and $5y$ being different, place them down one after another as at Art. 6. so is $3a - 5m - 21x + 5y$ the Sum required.

Exam. 3.

To	$-15a + 14m - 16$
Add	$7a - 14m + y$
Sum	$-8a - 16 + y$

Exam. 4.

$$\begin{array}{r} 11am - 7yd + mn \\ -5am - 2yd - 7a \\ \hline 6am - 9yd + mn - 7a \end{array}$$

Exam. 3. $-15a$ added to $7a$ is $-8a$, by Art. 3. and $14m$ added to $-14m$ is *nothing* or 0, by Art. 5. therefore take no Notice of those Quantities in the Sum, and -16 and y being different Quantities set them down by Art. 6. so is $-8a - 16 + y$ the Sum required.

Exam. 4. $11am$ added to $-5am$ is $6am$, by Art. 3. and $-7yd$ added to $-2yd$ is $-9yd$, by Art. 1. But mn and $-7a$ being different Quantities set them down by Art. 6. and $6am - 9yd + mn - 7a$ is the Sum required.

Exam. 5.

To	$4a - 17y + 15ap$
Add	$-2ap + 3a - 2y$
Sum	$13ap + 7a - 19y$

Exam. 6.

$$\begin{array}{r} -7m + 15 + 4a \\ -4a - 11 + 8m \\ \hline m + 4 \end{array}$$

Exam. 5. $-2ap$ added to $15ap$ is $13ap$, by Art. 3. and $3a$ added to $4a$ is $7a$, by Art. 1. and $-2y$ added to $-17y$ is $-19y$, by Art. 1. hence $13ap + 7a - 19y$ is the Sum required.

Exam. 6. $-7m$ added to $8m$, the Sum is m, by Art. 3. 15 added to -11, the Sum is 4, by Art. 3. and $4a$ added to $-4a$, the Sum is 0, or *nothing*, by Art. 5. hence $m + 4$ is the Sum required.

In these two Examples the same Quantities are not set under one another, to shew the Learner that however they are placed, if the Quantities are alike, they are to be added as if they stood one under the other.

The more perfectly Addition is understood, the easier it will render the Work of Substraction.

SUBSTRACTION,

SUBSTRACTION,

7. IS performed by one general Rule; change all the Signs of those Quantities which are to be substracted, or suppose them in the Mind to be changed, then add these Quantities to the others, according to the several Rules of Addition, which will be the Difference or Remainder required.

I would advise the Learner to take out the Examples, and put down those Quantities which are to be substracted with contrary Signs, to those they have in the Examples; that is, making those *affirmative* which are *negative*, and those *negative* which are *affirmative*, and then proceed as directed in the general Rule.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
From	$5a$	$7m$	$-5y$	$-8z$
Substract	$3a$	$2m$	$-2y$	$-4z$
Remains	$2a$	$5m$	$-3y$	$-4z$

Exam. 1. Here $3a$ the Quantity to be substracted has the Sign $+$, which being made or supposed to be made $-$, then by the general Rule, $5a$ is to be added to $-3a$, the Sum of which is $2a$, by Art. 3. and this is the Remainder required.

Exam. 2. In the same Manner $2m$ being supposed to have the Sign $-$ prefixed to it, then by the general Rule, $7m$ is to be added to $-2m$, the Sum of which is $5m$, by Art. 3. and this is the Remainder required.

Exam. 3 And if we suppose $-2y$ to be $2y$, or $+2y$, then by the general Rule, $-5y$ added to $+2y$, the Sum is $-3y$, by Art. 3. and this is the Remainder required.

Exam. 4. If we suppose $-4z$ to be $4z$, or $+4z$, then by the general Rule, if $-8z$ is added to $4z$, the Sum is $-4z$, by Art. 3. and this is the Remainder required.

	Exam. 5.	*Exam.* 6.	*Exam.* 7.	*Exam.* 8.
From	$14mn$	$-7yd$	$-5yx$	$4ay$
Substract	$-2mn$	$+5yd$	$+3yx$	$-3ay$
Remains	$16mn$	$-12yd$	$-8yx$	$7ay$

Exam. 5. The Sign of $2mn$ being $-$, if we suppose it $+$, then by the general Rule, $14mn$ added to $2mn$, the Sum is $16mn$, by Art. 1. the Remainder required.

 Exam.

Exam. 6. If we suppose $5yd$ to be $-5yd$, then by the general Rule, $-7yd$ added to $-5yd$, the Sum is $-12yd$, by Art. 1. the Remainder required.

Exam. 7. By supposing $3yx$ to be $-3yx$, then by the general Rule, $-5yx$ added to $-3yx$, the Sum is $-8yx$, by Art. 1. the Remainder required.

Exam. 8. And if we suppose $-3ay$ to be $3ay$, then by the general Rule, $4ay$ added to $3ay$, the Sum is $7ay$, by Art 1. the Remainder required.

From	$5am$	$-ay$	$-7ad$	$5yd$
Substract	$-am$	$-5ay$	$+ad$	yd
Remains	$6am$	$4ay$	$-8ad$	$4yd$

The Truth of Substraction may be proved as in common Arithmetic, by adding the Remainder to the Quantity which is substracted, and if their Sum is the same as that from which the Quantity was substracted, the Work is true, otherwise it is erroneous.

Thus in the four last Examples $6am$ added to $-am$, the Sum is $5am$.

And $4ay$ added to $-5ay$, the Sum is $-ay$.

And $-8ad$ added to ad, the Sum is $-7ad$.

And $4yd$ added to yd, the Sum is $5yd$. And in the same Manner may the other Examples be proved.

8. If two or more Quantities connected by the Signs $+$ or $-$, are to be substracted from other like Quantities connected by the Signs $+$ or $-$, it is done in the same Manner, only taking due Care to connect the remaining Quantities with their proper Signs, as was done in the Addition of compound Quantities.

	Exam. 9.	*Exam.* 10.	*Exam.* 11.
From	$12a+7b$	$7ma+5y$	$-5zy-2am$
Take	$3a+2b$	$6ma+4y$	$3zy+4am$
Remains	$9a+5b$	$ma+y$	$-8zy-6am$

Exam. 9. By supposing $3a$ to be $-3a$, then $-3a$ added to $12a$ the Sum is $9a$, by Art. 3. and again, supposing $2b$ to be $-2b$, then $-2b$ added to $7b$ the Sum is $5b$ by the same, and connecting these Quantities we have $9a+5b$, the Remainder required.

Exam. 10. $6ma$ being supposed negative, or to be $-6ma$, then $-6ma$ added to $7ma$ the Sum is ma, and $4y$ being

being ſuppoſed to be $-4y$, then $-4y$ added to $5y$ the Sum is y, hence $ma+y$ is the Remainder required.

Exam. 11. $3zy$ being ſuppoſed to be $-3zy$, and adding this to $-5zy$ the Sum is $-8zy$, by Art. 1. and $4am$ being ſuppoſed to be $-4am$, by adding that to $-2am$ the Sum by Art. 1. is $-6am$, hence $-8zy-6am$ is the Remainder required.

	Exam. 12.	*Exam.* 13.	*Exam.* 14.
From	$14a-5y$	$-7mn+2yd$	$2a+m$
Take	$-3a-5y$	$-3mn+3yd$	$5a-7$
Remains	$17a$	$-4mn-yd$	$-3a+m+7$

Exam. 12. The $-3a$ being ſuppoſed by the general Rule to be $3a$, and adding that to $14a$ the Sum is $17a$, by Art. 1. and the $-5y$ being ſuppoſed to be $5y$, if we add $5y$ to $-5y$, the Sum is a *Cypher*, or *nothing*, by Art. 5. hence $17a$ is the Remainder required.

Exam. 13. The $-3mn$ being ſuppoſed to be $3mn$, then by adding $3mn$ to $-7mn$ the Sum is $-4mn$, by Art. 3. and $3yd$ being ſuppoſed to be $-3yd$, and adding $-3yd$ to $2yd$ the Sum is $-yd$, by Art. 3. hence $-4mn-yd$ is the Remainder required.

Exam. 14. The $5a$ being ſuppoſed to be $-5a$, if that is added to $2a$ the Sum is $-3a$, by Art. 3. but the m and 7 being different Quantities, ſet them down by Art. 6. only take particular Care to change the Sign of 7, according to the general Rule for Subſtraction, then will $-3a+m+7$ be the Remainder required.

	Exam. 15.	*Exam.* 16.	*Exam.* 17.
From	$-am+y$	$15yd+20$	$14d+7-a$
Subſtract	$+am+y$	$-3yd-16$	$-d+7-8a$
Remains	$-2am$	$18yd+36$	$15d+7a$

The Truth of theſe Operations is proved in the ſame Manner as in Subſtraction of ſimple Quantities, by adding the Remainder to the Quantity which is ſubſtracted, and obſerving if that Sum is the ſame, and has the ſame Signs, with thoſe Quantities from which the Subſtraction was made. Thus,

Exam. 15. $-2am$ added to am, the Sum is $-am$, by Art. 3. to which connecting y with the Sign $+$, we find that by adding $-2am$ to $am+y$, the Sum is $-am+y$, the Quantity from which the Subſtraction was made.

Exam. 16. If 18 yd is added to — 3 yd the Sum is 15 yd, by Art. 3. and 36 added to — 16 the Sum is 20 by the ſame, hence the Sum of 18 yd + 36 added to — 3 yd — 16 is 15 yd + 20, the Quantity from which the Subſtraction was made.

Exam. 17. By adding 15 d to — d the Sum is 14 d, by Art. 3. and by adding 7 a to — 8 a the Sum is — a, by the ſame, to which putting down the 7, there being no Quantity to be added to that, hence 15 d + 7 a added to — d + 7 — 8 a the Sum is 14 d + 7 — a, the Quantity from which the Subſtraction was made.

But if the Quantities to be ſubſtracted are unlike thoſe from which the Subſtraction is to be made, ſet down theſe with the ſame Signs and Co-efficients they have in the Example, after which place the Quantities to be ſubſtracted with their Co-efficients, but change their Signs.

	Exam. 18.	*Exam.* 19.	*Exam.* 20.
From	$2a$	$-3y$	$5m$
Take	d	$2a$	$-2y$
Remains	$2a-d$	$-3y-2a$	$5m+2y$

Exam. 18. Having put down 2 a, after which put — d, the Quantity to be ſubſtracted being + d, and 2 a — d is the Remainder required.

Exam. 19. Having put down — 3 y, to that connect 2 a with the Sign —, ſo is — 3 y — 2 a the Remainder required. The 20th Example is done in the ſame Manner.

And if compound Quantities are to be ſubſtracted from compound Quantities, but unlike, ſet down all the Quantities one after the other, but change the Signs of thoſe Quantities which are to be ſubſtracted, as in theſe Examples.

From	$2a+5m$	$-4d+2p$
Take	$3y-2d$	$-5a+3y$
Remains	$2a+5m-3y+2d$	$-4d+2p+5a-3y$

Having wrote down 2 a + 5 m, to that connect 3 y with the Sign —, it being + in the Example, to which connect 2 d with the Sign +, it being — in the Example.

In the other Example, having wrote down — 4 d + 2 p, to this connect 5 a with the Sign +, it being — in the Example, to which connect 3 y with the Sign —, it being + in the Example.

MULTI-

MULTIPLICATION,

In which there are three Cases.

(9.) *Case* 1. WHEN the Signs of the Quantities to be multiplied, are both *affirmative*, or both *negative*, set or join the Letters together, and to them prefix the Sign +, which will be the Product required.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Multiply	a	y	$-z$	$-da$
By	d	m	$-a$	$-x$
Product	da	my	az	dax

Exam. 1. Having joined the Letters $d\,a$, and each of them having the *affirmative* Sign, therefore, by the Rule, $d\,a$, or $+\,d\,a$, is the Product required.

Exam. 2. Having joined the Letters m and y, and each of them having the affirmative Sign, therefore, by the Rule, $m\,y$, or $+\,m\,y$, is the Product required.

Exam. 3. Having joined the Quantities z and a, and each of them having the same Sign, therefore, by the Rule, $a\,z$, or $+\,a\,z$, is the Product required.

Exam. 4. Having joined the Quantities $d\,a$ and x, and both having the same Sign, therefore $d\,a\,x$, or $+\,d\,a\,x$, is the Product required.

	Exam. 5.	*Exam.* 6.	*Exam.* 7.	*Exam.* 8.
Multiply	a	$-am$	$-y$	am
By	a	$-d$	$-dp$	an
Product	aa	amd	dpy	$aman$

Exam. 5. Having joined $a\,a$, and both the Quantities being affirmative, therefore $a\,a$ is the Product required.

Exam. 6. Having joined the Quantities $a\,m$ and d, and both having the Sign —, hence $a\,m\,d$ is the Product required.

Exam. 7. Having joined the Quantities y and $d\,p$, and because both have the same Sign, therefore $d\,p\,y$ is the Product required.

Exam. 8. Having joined the Quantities $a\,m$ and $a\,n$, and both having the same Sign, therefore $a\,m\,a\,n$ is the Product required.

10. If

10. If the Multiplicand consists of two or more Quantities connected by the Signs $+$ or $-$, then the Multiplier must be multiplied into each of those Quantities, prefixing to each particular Multiplication its proper Sign, which will give the Product. Thus,

	Exam. 9.	*Exam.* 10.	*Exam.* 11.
Multiply	$a + d$	$z + y$	$- mx - n$
By	m	d	$- p$
Product	$ma + md$	$dz + dy$	$pmx + pn$

Exam. 9. If we multiply a by m, the Product is ma, by Art. 9. and multiplying d by m, the Product is md, by Art. 9. but as m and d have both the *affirmative* Sign, therefore prefix the Sign $+$ before md, and $ma + md$ is the Product required.

Exam. 10. Multiplying z by d, the Product is dz, and multiplying y by d, the Product is dy, by Art. 9. but as d and y have both the Sign $+$, prefixing that Sign before dy we have $dz + dy$, the Product required.

Exam. 11. Multiplying $- mx$ by $- p$, the Product is pmx, by Art. 9. and for the same Reason $- n$ multiplied by $- p$, the Product is pn, then connecting pmx and pn with the Sign $+$, we have $pmx + pn$, the Product required.

	Exam. 12.	*Exam.* 13.	*Exam.* 14.
Multiply	$- m - y$	$- a - zy$	$ad + z$
By	$- d$	$- x$	y
Product	$dm + dy$	$ax + xzy$	$ady + yz$

Exam. 12. Multiplying $- m$ by $- d$, the Product is md, by what was said at Example 11, and multiplying $- d$ by $- y$, the Product is for the same Reason dy, and connecting dm and dy with the Sign $+$, we have $dm + dy$, the Product required.

Exam. 13. Multiplying $- a$ by $- x$, we have ax for the Product, as in the last Example, and from multiplying $- zy$ by $- x$, we have for the same Reason xzy for this Product, then connecting ax and xzy with the Sign $+$, we have $ax + xzy$, the Product required.

Exam. 14. Multiplying ad by y, the Product is ady, and multiplying z by y, the Product is yz; but as the Signs of y and z are both alike, therefore prefixing the Sign $+$ to yz, we have $ady + yz$, the Product required.

11. It

11. It may be proper to caution the Learner, that in Multiplication it is quite indifferent which Letter he places firſt, or laſt, for to multiply am by d, the Product is amd, or mda, or dma, or adm, or any of the different Poſitions in which three Letters can be placed; this will be more compleatly and fully underſtood when we come to apply the Science to the Solution of Problems: But, that the Learner may form ſome Idea of the Truth of this, ſuppoſe we were to multiply 3, 5, and 7 together, the Product will be the ſame in whatever Order theſe three Numbers are multiplied. Thus,

$$\begin{array}{r} 3 \\ \underline{5} \\ 15 \\ \underline{7} \\ 105 \end{array} \qquad \begin{array}{r} 7 \\ \underline{5} \\ 35 \\ \underline{3} \\ 105 \end{array} \qquad \begin{array}{r} 3 \\ \underline{7} \\ 21 \\ \underline{5} \\ 105 \end{array}$$

This Obſervation I adviſe the Learner to fix in his Mind, to prevent concluding he has done any of the following Examples erroneouſly, by happening to place the Letters different from what they are in the Book.

12. But if the Multiplier and Multiplicand conſiſts of two or more Quantities, then begin and multiply the Multiplicand by any one Quantity in the Multiplier, according to the Directions in Art. 9. and 10. after that multiply the Multiplicand by another Quantity in the Multiplier, and put this Product under the other, and continue doing this till the Multiplicand has been multiplied by every Quantity in the Multiplier; then under theſe Products draw a Line, and add them together by the ſeveral Caſes of Addition, and this will be the Product required.

Exam. 1.

Multiply $a+b$
By $m+n$

$ma+mb$ the Product of $a+b$ multiplied by m, by Art. 10.
$na+nb$ the Product of $a+b$, multiplied by n, by the ſame.

$ma+mb+na+nb$ the Sum by Art. 6. the Product required.

Multiply

Multiply $m + y$
By $a + d$

$am + ay$ the Product of $m + y$ multiplied by a, by Art. 10.

$md + yd$ the Product of $m + y$ multiplied by d, by the fame.

$am + ay + md + yd$ the Sum by Art. 6. the Product required.

Multiply $-a - d$
By $-m - z$

$ma + md$ the Product of $-a - d$ multiplied by $-m$, by Art. 10.

$az + dz$ the Product of $-a - d$ multiplied by $-z$, by the fame.

$ma + md + az + dz$ the Sum by Art. 6. the Product required.

Multiply $a + b$
By $a + b$

$aa + ab$ the Product of $a + b$ multiplied by a, by Art. 10.

$ab + bb$ the Product of $a + b$ multiplied by b, by the fame.

$aa + 2ab + bb$ the Product of $a + b$ multiplied by $a + b$.

Now in the Addition of the laft Example I obferve there is ab in each of the two Lines, and there being no Co-efficient prefixt, *Unity* or 1 being then always underftood to be the Co-efficient, hence 1 ab added to 1 ab is $2ab$; the other Quantities aa and bb are fet down as in the former Examples, therefore $aa + 2ab + bb$ is the Product required.

And in fuch Additions as thefe I recommend it to the Learner, before he begins to add, to examine the feveral Quantities, and fee if the Letters in any two of them are alike, and if they are to collect them into one Sum, according to Art. 1 and 3; remembering, that though the Letters which compofe the two Quantities are not in the fame Order in each; yet if they are but the fame Letters, and no more in one than there is in the other Quantity, they are the fame, and may be added by Art. 1 and 3.

The

The four following Examples are for the Exercise of the Learner.

Multiply	$a + b$	$a + y$
By	$m + y$	$a + y$
	$am + mb$	$aa + ay$
	$ay + by$	$ay + yy$
Product	$am + mb + ay + by$	$aa + 2ay + yy$
Multiply	$y + m$	$-a - d$
By	$y + m$	$-a - d$
	$yy + ym$	$aa + ad$
	$ym + mm$	$aa + dd$
Product	$yy + 2ym + mm$	$aa + 2ad + dd$

13. *Case* 2. If there are Co-efficients or prefixt Numbers, then multiply the Numbers as in common Arithmetic, and to their Products join the Products of the Quantities found by the last *Case*, Art. 9.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Multiply	$2a$	$5m$	$-7ad$	$-2y$
By	$3m$	$3y$	$-3m$	$-a$
Product	$6am$	$15my$	$21adm$	$2ay$

Exam. 1. The Product of the Co-efficients 2 and 3 is 6: the Product of a multiplied by m is am, the Signs being alike, joining these together it is $6am$, the Product required.

Exam. 2. The Product of 5 by 3 is 15: the Product of m by y is my, for the Signs are alike, and joining these together it is $15my$, the Product required.

Exam. 3. The Product of 7 by 3 is 21: the Product of ad by m is adm, the Signs being alike, and joining the 21 and adm it is $21adm$, the Product required.

Exam. 4. The Product of 2, and 1 the Co-efficient of a, is 2, to which joining ay, the Product of a and y, it is $2ay$, the Product required, for the Signs of $2y$ and a are alike, being both negative.

Multiply	$7am$	$6dz$	$-3yp$	$-2dz$
By	$2d$	$2a$	$-3y$	$-d$
Product	$14amd$	$12dza$	$9yyp$	$2ddz$

14. And

14. And if there are two or more Quantities with Co-efficients connected by the Signs + or —, to be multiplied by any Quantity and its Co-efficient, they are multiplied as in the last Article, only connecting the several particular Products together with their proper Signs, as was done at Art. 10.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
Multiply	$2a+3b$	$3y+5d$	$-2y-2a$
By	$3m$	$5m$	$-3z$
Product	$6am+9bm$	$15ym+25dm$	$6yz+6za$

Exam. 1. Multiplying $2a$ by $3m$ the Product is $6am$, by Art. 13. and then multiplying $3m$ by $3b$ the Product is $9bm$, by Art. 13. to which prefixing the *affirmative* Sign, as the Signs of $3b$ and $3m$ are alike, and $6am+9bm$ is the Product required.

Exam. 2. Multiplying $3y$ by $5m$ the Product is $15ym$, by Article 13. then multiplying $5m$ by $5d$ the Product is $25dm$, to which prefixing the affirmative Sign, as the Signs of $5d$ and $5m$ are alike, and $15ym+25dm$ is the Product required.

Exam. 3. Multiplying $-2y$ by $-3z$ the Product is $6yz$, by Art. 9 and 13. Again, multiplying $-2a$ by $-3z$ the Product is $6az$, for the Signs of $2a$ and $3z$ are alike, and connecting $6yz$ and $6za$ with the Sign +, we have $6yz+6za$, the Product required.

	Exam. 4.	*Exam.* 5.	*Exam.* 6.
Multiply	$3m+2y$	$-2d-3m$	$-3y-7my$
By	$6a$	$-4a$	$-2a$
Product	$18ma+12ya$	$8da+12ma$	$6ya+14amy$

Exam. 4. Multiplying $3m$ by $6a$ the Product is $18ma$, and multiplying $2y$ by $6a$ the Product is $12ya$, and placing the Sign + before $12ya$, because the Signs of $6a$ and $2y$ are alike, we have $18ma+12ya$, the Product required.

Exam. 5. Multiplying $-2d$ by $-4a$ the Product is $8da$, for the Signs of $2d$ and $4a$ are alike, being both negative, therefore $8da$ or $+8da$ is the Product of these Quantities. Now multiplying $-4a$ by $-3m$ the Product is $12ma$, to which prefixing the affirmative Sign, as $3m$ and $4a$ have the same Sign, both being negative, and we have $8da+12ma$, the Product required.

 Exam.

Exam. 6. The Product of $-3y$ by $-2a$ is $6ya$, or $+6ya$, and the Product of $-7my$ by $-2a$ is $14amy$, or $+14amy$, hence, for the Reason in the last Example, $6ya+14amy$ is the Product required.

Multiply	$-3m-2d$	$-2z-3y$	$-4d-5m$
By	$-4a$	$-4a$	$-2b$
Product	$12ma+8da$	$8za+12ay$	$8bd+10mb$

15. And if there are two or more Quantities with Co-efficients connected by the Signs + or —, to be multiplied by two or more Quantities with Co-efficients connected in the same Manner, the Quantities are to be multiplied as at Art. 12. taking due Care to multiply the Co-efficients, as has been taught Art. 14. Thus,

Multiply $2a+3b$
By $3a+5m$

$6aa+9ab$ the Product of $2a+3b$ multiplied by $3a$, by Art. 14.

$10ma+15bm$ the Product of $2a+3b$ multiplied by $5m$, by the same.

$6aa+9ab+10ma+15bm$ the Product required, being the Sum of the two particular Products, which are added together by Art. 6.

Multiply $3m+5y$
By $2a+3n$

$6am+10ay$ the Product of $3m+5y$ multiplied by $2a$, by Art. 14.

$9mn+15yn$ the Product of $3m+5y$ multiplied by $3n$, by Art. 14.

$6am+10ay+9mn+15yn$ the Product required, being the Sum of the two Products, which are added together by Art. 6.

Multiply $2a+3b$
By $2a+2b$

$4aa+6ab$ the Product of $2a+3b$ multiplied by $2a$, by Art. 14.

$4ab+6bb$ the Product of $2a+3b$ multiplied by $2b$, by the same.

$4aa+10ab+6bb$ the Product required. In this

Addition

Addition the Reader is to obſerve that in one Line there is $6ab$, and in the other Line there is $4ab$, which two Quantities added together the Sum is $10ab$, by Art. 1. but the $4aa$ and $6bb$ being different Quantities, they are ſet down by Art. 6. hence the Product of $2a+3b$ multiplied by $2a+2b$, is $4aa+10ab+6bb$.

Multiply	$3a+7b$	$3a+2b$
By	$2a+5n$	$a+4b$
	$6aa+14ba$	$3aa+2ba$
	$15an+35bn$	$12ba+8bb$
Product	$6aa+14ba+15an+35bn$	$3aa+14ba+8bb$

16. *Caſe* 3. When the Signs of the two Quantities to be multiplied are one affirmative and the other negative, then multiply the Quantities as before directed, but to their Product prefix —, or the negative Sign.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Multiply	$-b$	$-a$	$-am$	$-dm$
By	a	d	y	z
Product	$-ba$	$-ad$	$-amy$	$-dmz$

Exam. 1. The Product of b by a is ba, for Multiplication is only joining the Letters, but as the Sign of b is —, and that of a is +, therefore to ba prefix the Sign —, ſo is $-ba$ the Product required.

Exam. 2. The Product of a by d is ad, but as the Signs of a and d are different, therefore prefix the Sign — to ad, and $-ad$ is the Product required.

Exam. 3. The Product of am by y is amy, but as the Signs of am and y are different, therefore prefix the Sign — to amy, ſo is $-amy$ the Product required.

Exam. 4. The Product of dm by z is dmz, but as the Signs of dm and z are different, therefore prefix the Sign — to dmz, ſo is $-dmz$ the Product required.

This Operation being the ſame as at Art. 9. taking Care to make the Sign of the Product —, I ſhall only ſubjoin the following Examples for the Exerciſe of the Learner.

Multiply

Multiply	xm	$-ym$	$-ax$	ma
By	$-y$	d	dy	$-p$
Product	$-xmy$	$-ymd$	$-axdy$	$-map$

17. And if two or more Quantities with Co-efficients are to be multiplied into any one Quantity with a different Sign, they are multiplied as at Art. 14. taking Care of the Signs arising in the Product, according to Art. 9. and 16.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
Multiply	$-3a-2z$	$2y+5d$	$-5m-2y$
By	$3m$	$-3a$	$3d$
Product	$-9am-6zm$	$-6ay-15ad$	$-15md-6dy$

Exam. 1. Multiplying $3a$ by $3m$, the Product by Art. 13. is $9am$, but as $3m$ has the Sign $+$ prefixed to it, and $3a$ has the Sign $-$ prefixed to it, therefore to the Product $9am$ prefix the Sign $-$, by Art. 16. Again, $2z$ multiplied by $3m$ the Product is $6zm$, but as $2z$ has the Sign $-$ to it, and $3m$ the Sign $+$, therefore to $6zm$ prefix the Sign $-$, by Art. 16. and $-9am-6zm$ is the Product required.

Exam. 2. Multiplying $2y$ by $3a$, the Product is $6ay$, but as the Sign of $2y$ is $+$, and that of $3a$ is $-$, therefore to $6ay$ prefix the Sign $-$, by Art. 16. Then multiplying $5d$ by $3a$ the Product is $15ad$, but as the Sign of $5d$ is $+$, and that of $3a$ is $-$, therefore prefix the Sign $-$ to $15ad$ by Art. 16. and $-6ay-15ad$ is the Product required.

Exam. 3. Multiplying $5m$ by $3d$, the Product is $15md$, but as the Sign of $5m$ is $-$, and that of $3d$ is $+$, therefore prefix the Sign $-$ to $15md$. Again, multiplying $2y$ by $3d$, the Product is $6yd$, but because the Signs of $2y$ and $3d$ are different, therefore prefix the Sign $-$ to $6dy$, and $-15md-6dy$ is the Product required.

Examples for the Exercise of the Learner.

Multiply	$-2a-3b$	$3m+7y$	$-5y-3b$
By	$4z$	$-2d$	$3m$
Product	$-8az-12bz$	$-6md-14dy$	$-15ym-9bm$

18. And if two or more Quantities with Co-efficients are to be multiplied by two or more Quantities with Co-efficients, if their Signs are unlike, yet they are multiplied as at Art. 15. taking due Care of the Signs of the Product, by Art. 9. and 16.

Multiply $-3a-2m$
By $\quad 4b+6y$

$-12ab-8bm$ the Product of $-3a-2m$ multiplied by $4b$, by Art. 17.
$-18ay-12ym$ the Product of $-3a-2m$ multiplied by $6y$, by Art. 17.

$-12ab-8bm-18ay-12ym$ the Product required, being the Sum of the two particular Products, which are added by Art. 6.

Multiply $\quad 5y+3m$
By $-7d-3a$

$-35yd-21dm$ the Product of $5y+3m$ multiplied by $-7d$, by Art. 17.
$-15ay-9am$ the Product of $5y+3m$ multiplied by $-3a$, by Art. 17.

$-35yd-21dm-15ay-9am$ the Product required, being the Sum of the two particular Products, which are added by Art. 6.

Multiply $\quad 2a+3b$
By $-2a-3b$

$-4aa-6ab$ the Product of $2a+3b$ multiplied by $-2a$, by Art. 17.
$-6ab-9bb$ the Product of $2a+3b$ multiplied by $-3b$, by Art. 17.

$-4aa-12ab-9bb$ the Product required: For in this Addition the Reader may obſerve that there is $-6ab$ in each of the two particular Products, which being added together by Art. 1. make $-12ab$, but $-4aa$ and $-9bb$ being different Quantities, they muſt be placed ſeparate from one another. There are Examples of this Kind at Art. 15.

19. It may be for the Learner's Advantage to be put in Mind, that if any Algebraic Quantities are to be multiplied by a pure Number, that then this Number is to be multiplied into every one of the Co-efficients of the other Quantities, in all Reſpects as before, and to each particular Product ſet or join that Quantity whoſe Co-efficient was multiplied. Thus,

Exam.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
Multiply	$2a + 3b$	$-3m - 4d$	$4d + 3y$
By	6	7	9
Product	$12a + 18b$	$-21m - 28d$	$36d + 27y$

Exam. 1. Multiplying 6 by 2 the Product is 12, to which joining a it is $12a$, then multiplying 3 by 6 it is 18, to which joining b it is $18b$, and becauſe 6 and $3b$ have both the Sign $+$, therefore by Art. 9. prefix the Sign $+$ to $18b$, ſo is $12a + 18b$ the Product required.

Exam. 2. Multiplying 3 by 7 it is 21, to which joining m it is $21m$, but as the Sign of $3m$ is $-$, and that of 7 is $+$, therefore by Art. 16. prefix the Sign $-$ to $21m$, and it is $-21m$. Again, multiplying 4 by 7 it is 28, to which joining d it is $28d$, but as the Signs of $4d$ and 7 are likewiſe unlike, therefore to $28d$ prefix the Sign $-$, and $-21m - 28d$ is the Product required.

Exam. 3. Multiplying 4 by 9 the Product is 36, to which joining d it is $36d$, and becauſe the Signs of $4d$ and 9 are alike, therefore it will be $36d$, or $+36d$; and multiplying 9 by $3y$ the Product will be $27y$, to which muſt be prefixt the Sign $+$, becauſe $3y$ and 9 have the ſame Sign, ſo is $36d + 27y$ the Product required.

Examples wherein all the three *Caſes* of Multiplication are promiſcuouſly uſed.

$$\begin{array}{ll}
\text{Multiply} & 2a - 3b \\
\text{By} & 5m + 2y \\
\hline
 & 10ma - 15mb \\
 & \quad 4ay - 6yb \\
\hline
\text{Product} & 10ma - 15mb + 4ay - 6yb
\end{array}$$

The $2a$ being multiplied by $5m$ the Product is $10ma$, by Art. 13. and $-3b$ being multiplied by the ſame $5m$, the Product is $-15mb$, by Art. 16. and 17.

And $2a$ being multiplied by $2y$ the Product is $4ay$, by Art. 13. and $-3b$ multiplied by $2y$ the Product is $-6yb$, by Art. 16. and 17.

Now draw the Line and begin to add them, and becauſe the Quantities are all different, they are added by Art. 6. and therefore the Product will be $10ma - 15mb + 4ay - 6yb$.

Multiply	$-7m+2a$
By	$3y-4n$
	$-21my+6ay$
	$28mn-8na$
Product	$-21my+6ay+28mn-8na$

The $-7m$ multiplied by $3y$ the Product is $-21my$, by Art. 16 and 17. and $2a$ multiplied by $3y$ the Product is $6ay$, by Art. 13.

And $-7m$ multiplied into $-4n$ the Product is $28mn$, by Art. 13. and $-4n$ multiplied by $2a$ the Product is $-8na$, by Art. 16 and 17.

Now begin the Addition, and becaufe the Quantities are all different, they are added by Art. 6. and the Product is $-21my+6ay+28mn-8na$.

Multiply	$2a+3b$
By	$2a-3b$
	$4aa+6ab$
	$-6ab-9bb$
Product	$4aa-9bb$

Multiplying $2a$ by $2a$ the Product is $4aa$, and multiplying $3b$ by $2a$ the Product is $6ab$.

And multiplying $2a$ by $-3b$ the Product is $-6ab$, becaufe the Signs of the two Quantities are unlike, and for the fame Reafon the Product of $3b$ by $-3b$ is $-9bb$.

Now begin the Addition, and I obferve in the firft Line there is $+6ab$ or $6ab$, but in the fecond Line there is $-6ab$, now becaufe the Co-efficients are equal, the Quantities alike, but the Signs being contrary, therefore by Art. 5. thefe Quantities will deftroy one another, then putting down the $4aa$ and $-9bb$, by Art. 6. we have $4aa-9bb$, the Product required.

Multiply	$7m+4a$
By	$-3a+5$
	$-21ma-12aa$ the Product of $7m+4a$ multiplied by $-3a$.
	$35m+20a$ the Product of $7m+4a$ multiplied by 5, by Art. 19.
Product	$-21ma-12aa+35m+20a$

Multiply

Multiply	$5a + b$	
By	$2a + 3b$	
	$10aa + 2ab$	the Product of $5a + b$ multiplied by $2a$.
	$15ab + 3bb$	the Product of $5a + b$ multiplied by $3b$.
Product	$10aa + 17ab + 3bb$.	

DIVISION,

In which there are four Cases.

20. *Case* 1. WHEN the Signs of the Quantities to be divided are both affirmative, or both negative, reject all those Quantities in the Dividend and Divisor that are alike, and set down the Remainder, prefixing to it the Sign +, which will be the Quotient required.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Divide	ab	dm	$-mn$	$-ap$
By	a	d	$-m$	$-p$
Quotient	b	m	n	a

Exam. 1. Because a is in the Dividend and Divisor, reject it, and b being only left, it is the Quotient sought, and is to have the Sign +, because the Signs of ab and a are alike.

Exam. 2. Because d is in the Dividend and Divisor, reject it, and there being only m left, it is the Quotient sought, which must have the Sign +, because the Signs of dm and d are alike.

Exam. 3. Because m is in the Dividend and Divisor, reject it, and n being only left, I write it down for the Quotient sought, which must have the Sign +, because mn and m have the same Sign.

Exam. 4. Because p is in the Dividend and Divisor, I reject it, and place down a, the Quantity left, for the Quotient sought, which must have the Sign +, for the Signs of ap and p are alike.

Exam.

	Exam. 5.	*Exam.* 6.	*Exam.* 7.	*Exam.* 8.
Divide	amd	$-apy$	mda	$-myz$
By	am	$-py$	ma	$-z$
Quotient	d	a	d	ym

Exam. 5. Becauſe am is in the Dividend and Diviſor, reject it, and place down d for the Quotient, which muſt have the affirmative Sign, for the Signs of amd and am are alike.

Exam. 6. Becauſe py is in the Dividend and Diviſor, I reject it, and place down a, the remaining Quantity, for the Quotient, which muſt have the affirmative Sign, for the Signs of apy and py are alike.

Exam. 7. Becauſe ma is in the Dividend and Diviſor, I reject it, and place down d for the Quotient, which muſt have the Sign $+$, becauſe the Signs of mda and ma are alike.

Exam. 8. Becauſe z is in the Dividend and Diviſor, I reject it, and place down ym, or my, which is the ſame thing, for the Quotient ſought, and muſt have the Sign $+$, becauſe the Signs of myz and z are alike.

Divide	apz	$-mnd$	$-abc$	$abdy$
By	az	$-md$	$-c$	ay
Quotient	p	n	ab	bd

The Truth of theſe Operations in Diviſion may be proved like thoſe in Arithmetic, for the Quotient and Diviſor being multiplied, the Product will be the Dividend if the Work is true; thus in the ſecond Example of the laſt four, by multiplying n the Quotient into $-md$ the Diviſor, the Product is mdn, or mnd, to which muſt be prefixt the Sign $-$, by Art. 16. becauſe the Signs of md and n are unlike, hence the Product with its Sign is $-mnd$, the given Dividend.

And in the laſt Example, if we multiply bd the Quotient by ay the Diviſor, the Product is $bday$, or $abdy$, which is the ſame thing, by Art. 11. and this Quantity muſt have the affirmative Sign, by Art. 9. for the Signs of bd and ay are alike, hence $+abdy$, or $abdy$, is the Product with its Sign, the ſame as the given Dividend: And ſo of any other Example.

21. But if all the Quantities in the Diviſor are not to be found in the Dividend, then you muſt only reject thoſe Quantities in the

the Dividend and Divisor that are alike, placing down the remaining Quantities of the Dividend, and under them those of the Divisor that are not rejected by this Rule; which will be the Quotient sought, and stand like a Vulgar Fraction in common Arithmetic.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Divide	amb	$-mndz$	$-dayp$	$pnqr$
By	ay	$-mna$	$-dpz$	pad
Quotient	$\frac{mb}{y}$	$\frac{dz}{a}$	$\frac{ay}{z}$	$\frac{nqr}{ad}$

Exam. 1. Because a is in the Dividend and Divisor, reject it, and place down mb the remaining Part of the Dividend, under which drawing a Line, and place y the remaining Part of the Divisor, so will $\frac{mb}{y}$ be the Quotient sought, which must have the Sign $+$, by Art. 20. as the Signs of the Quantities to be divided are alike.

Exam. 2. Because mn is in the Dividend and Divisor, reject it, and place down dz the remaining Part of the Dividend, under which drawing a Line, and place a the remaining Part of the Divisor, so is $\frac{dz}{a}$ the Quotient required, and it must have the Sign $+$, by Art. 20. as the Signs of the Quantities to be divided are alike.

Exam. 3. Because dp is in both Dividend and Divisor, reject it, and write down ay the remaining Part of the Dividend, under which place z the remaining Part of the Divisor, as in the two former Examples, so is $\frac{ay}{z}$, or $+\frac{ay}{z}$, the Quotient required, for the Signs of the two Quantities to be divided are alike.

Exam. 4. Because p is in both Dividend and Divisor, reject it, and write down nqr the remaining Part of the Dividend, under which place ad the remaining Part of the Divisor, and $\frac{nqr}{ad}$ is the Quotient required, which will be affirmative by Art. 20. because the Signs of $pnqr$ and pad are alike.

Divide	$-apqn$	adz	mnd	$-yzdb$
By	$-anm$	ap	ma	$-yza$
Quotient	$\frac{pq}{m}$	$\frac{dz}{p}$	$\frac{nd}{a}$	$\frac{db}{a}$

These Operations are proved as at Art. 20. by multiplying the Quotient by the Divifor; for in the laft Example the Quotient is $\frac{db}{a}$, which is a Fraction: the Divifor is $-yza$, which by the Rule of Vulgar Fractions in common Arithmetic is made this improper Fraction $-\frac{yza}{1}$, then the two Fractions to be multiplied are $\frac{db}{a}$, and $-\frac{yza}{1}$, multiplying the Numerators we have $dbyza$ for the new Numerator, and multiplying the Denominators we have a for the Denominator, hence the Product is this Fraction $\frac{dbyza}{a}$, but as $\frac{yza}{1}$ has the *negative* Sign, and $\frac{db}{a}$ has the affirmative Sign, therefore by Art. 16. prefix the Sign $-$ to $\frac{dbyza}{a}$ and it is $-\frac{dbyza}{a}$, the Product with its true Sign: But in this Fraction as $-dbyza$ is to be divided by a, rejecting a both in Dividend and Divifor by Art. 20. we have $-dbyz$ or $-yzdb$, the fame with the Dividend in the given Example; in like Manner may the others be proved.

22. And if there are two or more Quantities connected by the Signs $+$ or $-$ to be divided by any fingle Quantity, every Quantity in the Dividend muft be divided by the Divifor, fetting down the particular Quotients, as at Art. 20. which muft be connected by the Sign $+$ when the Signs of the Quantities to be divided are both alike.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
Divide	$ab+am$	$md+mz$	$-da-dpq$
By	a	m	$-d$
Quotient	$b+m$	$d+z$	$a+pq$

Exam. 1. Dividing ab by a the Quotient is b, by Art. 20. and dividing am by a the Quotient is m, by the fame Art. but as

as $a m$ and a have both the affirmative Sign, therefore to m prefix the Sign $+$, so is $b + m$ the Quotient required.

Exam. 2. $m d$ being divided by m the Quotient is d, by Art. 20. and dividing $m z$ by m the Quotient is z, to which prefixing the Sign $+$, as $m z$ and m have both the same Sign, we have $d + z$ for the Quotient required.

Exam. 3. $d a$ being divided by d the Quotient is a, and because $d a$ and a have both the negative Sign, or the Signs are alike, therefore a must have the Sign $+$, whence it is $+ a$ or a, and dividing $- d p q$ by $- d$ the Quotient is $p q$, to which must be prefixed the Sign $+$, for the Signs of $d p q$ and d are alike, hence we have $a + p q$ for the Quotient required.

	Exam. 4.	*Exam.* 5.	*Exam.* 6.
Divide	$- a b - a m$	$b m + b n$	$- z y p - z y a$
By	$- a$	b	$- z y$
Quotient	$b + m$	$m + n$	$p + a$

Exam. 4. Dividing $- a b$ by $- a$ the Quotient is b, by Art. 20. and it must be $+ b$ or b, as the Signs of $a b$ and a are alike: then dividing $- a m$ by $- a$ the Quotient is m, by Art. 20. because the Signs of $a m$ and a are alike, then connecting b and m with the Sign $+$, and $b + m$ is the Quotient required.

Exam. 5. Dividing $b m$ by b the Quotient is m, by Art. 20. and dividing $b n$ by b the Quotient is n, and as $b n$ and b have the same Sign, therefore prefix the Sign $+$ to n, so is $m + n$ the Quotient required.

Exam. 6. Dividing $- z y p$ by $- z y$ the Quotient is p, by Art. 20. and dividing $- z y a$ by $- z y$ the Quotient is a, to which prefixing the Sign $+$, for the Signs of $z y a$ and $z y$ are alike, we have $p + a$ the Quotient required.

Divide	$- d n z - z a d$	$a m + a d$	$- d y - d z$
By	$- d z$	a	$- d$
Quotient	$n + a$	$m + d$	$y + z$

The Truth of these Operations are proved by multiplying the Quotient by the Divisor; if that Product agrees with the Dividend in its Quantities and Signs the Work is true, otherwise not. Now in the last Example the Quotient $y + z$, and the Divisor $- d$, being multiplied together by Art. 14. they produce $- d y - d x$, the given Dividend.

23. *Case* 2. When the Signs of the Quantities to be divided are one *affirmative* and the other *negative*, find the Quotient of the Quantities as before; but to them prefix the *negative* Sign, or Sign —.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Divide	$-am$	$-mnp$	ayz	dmb
By	a	m	$-ay$	$-db$
Quotient	$-m$	$-np$	$-z$	$-m$

Exam. 1. Because a is in both Dividend and Divisor, reject it, and place down m the remaining Part of the Dividend, but as the Signs of am and a are different, therefore to m prefix the Sign —, and it will be $-m$, the Quotient required.

Exam. 2. Because m is in the Dividend and Divisor, reject it, and place down np the remaining Part of the Dividend, but as the Signs of mnp and m are different, therefore to np prefix the Sign —, and it will be $-np$, the Quotient required.

Exam. 3. Because ay is in the Dividend and Divisor, reject it, and place down z the remaining Part of the Dividend, but as the Signs of ayz and ay are different, prefix the Sign — to z, and it will be $-z$, the Quotient required.

Exam. 4. Because db is in the Dividend and Divisor, reject it, and place down m the remaining Part of the Dividend; but the Signs of the Quantities that are divided being different, to m prefix the Sign —, and it will be $-m$, the Quotient required.

Divide	$-mnp$	$-mnp$	dyp	dab
By	m	mn	$-dy$	$-db$
Quotient	$-np$	$-p$	$-p$	$-a$

24. And if there are two or more Quantities connected by the Signs + or —, to be divided by any single Quantity, the Operation is the same as at Art. 22. only taking due Care, when the Signs of the Quantities to be divided are different, to prefix the Sign — before those Quotients.

Exam.

	Exam. 1.	Exam. 2.	Exam. 3.
Divide	$-mn-md$	$ad+ab$	$-dnz-dzy$
By	m	$-a$	dz
Quotient	$-n-d$	$-d-b$	$-n-y.$

Exam. 1. Dividing $-mn$ by m the Quotient is n, by Art. 20. but as the Signs of mn and m are different, therefore by Art. 23. I prefix the Sign $-$ to n, and it is $-n$. And dividing $-md$ by m, the Quotient is d; but as the Signs of md and m are different, therefore by Art. 23. prefix the Sign $-$ to d, hence $-n-d$ is the Quotient required.

Exam. 2. Dividing ad by $-a$ the Quotient is d, to which prefix the Sign $-$, by Art. 23. which makes it $-d$: then dividing ab by $-a$, the Quotient is b; but as the Signs of ab and a are different, therefore by Art. 23. prefix the Sign $-$ to b, and $-d-b$ is the Quotient required.

Exam. 3. Dividing $-dnz$ by dz the Quotient is $-n$, by Art. 20. and 23. and for the fame Reafon dividing $-dzy$ by dz, the Quotient is $-y$, which placing after $-n$, we have $-n-y$, the Quotient required.

Divide	$maz+mzd$	$-dab-dby$	$-azx-azb$
By	$-mz$	db	az
Quotient	$-a-d$	$-a-y$	$-x-b$

The Truth of thefe Operations are likewife proved from multiplying the Quotient by the Divifor, and if it agrees with the Dividend in its Quantities and Signs, the Work is true, otherwife not.

25. *Cafe* 3. But when there are Co-efficients joined to the Quantities, divide the Co-efficients as in common Arithmetic; and to their Quotients join the Quotient of the Quantities found by the foregoing Directions; but cautioufly remember, that if the Signs of the Quantities that are divided are alike, the Quotient muft have the affirmative Sign, as at Art. 20. but if the Signs of the Quantities that are divided are unlike, then the Quotient muft have the Sign $-$ prefixt to it, by Art. 23.

	Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
Divide	$16am$	$8yz$	$-24dm$	$-18ma$
By	$2a$	$2z$	$-6d$	$-6a$
Quotient	$8m$	$4y$	$4m$	$3m$

Exam.

Exam. 1. Dividing 16 by 2 the Quotient is 8, and am divided by a the Quotient is m, joining 8 to the m it is $8m$, and as $16am$ and $2a$ have the fame Sign, hence by Art. 20. the Sign + muft be prefixt to $8m$, therefore the Quotient is $+8m$ or $8m$.

Exam. 2. Dividing 8 by 2 the Quotient is 4, and dividing yz by z the Quotient is y, joining the 4 to the y it is $4y$; but as $8yz$ and $2z$ have the fame Sign, therefore by Art. 20. prefix the Sign + to $4y$, hence $+4y$ or $4y$ is the Quotient required.

Exam. 3. Dividing 24 by 6 the Quotient is 4, and dividing dm by d the Quotient is m, joining 4 to the m it is $4m$; but as $24dm$ and $6d$ have the fame Sign, prefix the Sign + to $4m$, hence $+4m$ or $4m$ is the Quotient required.

Exam. 4. Dividing 18 by 6 the Quotient is 3, and dividing ma by a the Quotient is m, joining 3 to the m it is $3m$, and as $18ma$ and $6a$ have the fame Sign, therefore by Art. 20. the Quotient is $+3m$ or $3m$.

	Exam. 5.	*Exam.* 6.	*Exam.* 7.	*Exam.* 8.
Divide	$-15ay$	$-8dm$	$28yz$	$-12da$
By	$3a$	$-4d$	$-7y$	$3a$
Quotient	$-5y$	$2m$	$-4z$	$-4d$

Exam. 5. Dividing 15 by 3 the Quotient is 5, and dividing ay by a the Quotient is y, joining 5 to the y it is $5y$, but as the Signs of $15ay$ and $3a$ are different, therefore by Art. 23. prefix the Sign — to $5y$, and $-5y$ is the Quotient required.

Exam. 6. Dividing 8 by 4 the Quotient is 2, and dividing dm by d the Quotient is m, joining the 2 and m it is $2m$; but as $8dm$ and $4d$ have the fame Sign, prefix the Sign + to $2m$, by Art. 20. and $+2m$ or $2m$ is the Quotient required.

Exam. 7. Dividing 28 by 7 the Quotient is 4, and dividing yz by y the Quotient is z, joining the 4 and z it is $4z$; but as $28yz$ and $7y$ have different Signs, therefore by Art. 23. prefix the Sign — to $4z$, fo will $-4z$ be the Quotient required.

Exam. 8. Dividing 12 by 3 the Quotient is 4, and dividing da by a the Quotient is d, joining the 4 and d it is $4d$; but as the Signs of $12da$ and $3a$ are different, therefore by Art. 23. prefix the Sign — to $4d$, and then $-4d$ is the Quotient required.

Divide

Divide	$-32\,am$	$18\,dza$	$-22\,ymn$	$16\,az$
By	$-8\,m$	$9\,a$	$11\,yn$	$-8\,z$
Quotient	$4\,a$	$2\,dz$	$-2\,m$	$-2\,a$

26. And if there are two or more Quantities connected together with Co-efficients, to be divided by any single Quantity and its Co-efficient, the Operation is still performed in the same Manner, connecting the particular Quotients as at Art. 22, and 24. still carefully remembering that when the Quantities that are divided have like Signs, whether *affirmative* or *negative*, the Quotient must have the *affirmative* Sign; but if the Signs of the Quantities that are divided are unlike, then the Quotient must have the Sign — prefixt to it.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
Divide	$4\,am + 12\,ad$	$-16\,my + 24\,mz$	$28\,dn - 21\,db$
By	$2\,a$	$-4\,m$	$7\,d$
Quotient	$2\,m + 6\,d$	$4\,y - 6\,z$	$4\,n - 3\,b$

Exam. 1. Dividing $4\,am$ by $2\,a$ the Quotient is $2\,m$, by Art. 25. and dividing $12\,ad$ by $2\,a$ the Quotient is $6\,d$, and because the Signs of $2\,a$ and $12\,ad$ are alike, prefix the Sign $+$ to $6\,d$, and we have $2\,m + 6\,d$, the Quotient required.

Exam. 2. Dividing $-16\,my$ by $-4\,m$ the Quotient is $4\,y$, by Art. 25. for the Signs of $16\,my$ and $4\,m$ are alike, and dividing $24\,mz$ by $-4\,m$ the Quotient is $-6\,z$, for $6\,z$ must have the negative Sign prefixt to it, the Signs of $24\,mz$ and $4\,m$ being unlike; hence $4\,y - 6\,z$ is the Quotient required.

Exam. 3. Dividing $28\,dn$ by $7\,d$ the Quotient is $4\,n$, or $+4\,n$, for the Signs of $28\,dn$ and $7\,d$ are alike: and dividing $-21\,db$ by $7\,d$ the Quotient is $-3\,b$, for $3\,b$ must have the negative Sign prefixt to it, as the Signs of $21\,db$ and $7\,d$ are unlike, hence $4\,n - 3\,b$ is the Quotient required.

Divide	$16\,pa - 28\,pd$	$-24\,nm + 36\,mz$	$16\,zu - 4\,zd$
By	$-4\,p$	$-4\,m$	$2\,z$
Quotient	$-4\,a + 7\,d$	$6\,n - 9\,z$	$8\,u - 2\,d$

The Truth of these Operations are proved likewise from multiplying the Quotient by the Divisor, for if the Work is true, the Product will agree with the Dividend in its Quantities and Signs: In the last Example the Divisor is $2\,z$, and the Quotient is $8\,u - 2\,d$, now if we

Multiply

Multiply	$8u - 2d$
By	$2z$
	$16zu - 4zd$

the Product is the same as the given Dividend, and so may the other Examples be proved.

27. *Case* 4. But when any Quantities in the Dividend are not the same with those in the Divisor, then place down the Dividend with its Signs and Co-efficients, under which drawing a Line, and after the Manner of Vulgar Fractions place the Divisor with its Signs and Co-efficients, and this will be the Quotient required.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Divide	b	am	$3my$	$2dy$
By	a	d	z	b
Quotient	$\frac{b}{a}$	$\frac{am}{d}$	$\frac{3my}{z}$	$\frac{2dy}{b}$

Exam. 1. Because b and a are different Quantities, place down the Dividend b, under which draw a Line, and place the Divisor a, so is $\frac{b}{a}$ the Quotient required.

Exam. 2. Because am and d are different Quantities, place down am the Dividend, draw a Line under it, and place the Divisor d, so is $\frac{am}{d}$ the Quotient required.

Exam. 3. Because $3my$ and z are different Quantities, place down $3my$ the Dividend, under it draw a Line, and place z the Divisor, so is $\frac{3my}{z}$ the Quotient required.

Exam. 4. Because $2dy$ and b are different Quantities, place down $2dy$ the Dividend, draw a Line under it, and place b the Divisor, and $\frac{2dy}{b}$ is the Quotient required.

Divide	$2ma$	$5dz$	$21ma$	$8yd$
By	$3y$	$2y$	$5d$	$3z$
Quotient	$\frac{2ma}{3y}$	$\frac{5dz}{2y}$	$\frac{21ma}{5d}$	$\frac{8yd}{3z}$

Divide	$m a$	$7 d$	$3 m b c$	$24 d$
By	$7 y$	$m z$	$y d$	$7 y z$
Quotient	$\frac{m a}{7 y}$	$\frac{7 d}{m z}$	$\frac{3 m b c}{y d}$	$\frac{24 d}{7 y z}$

28. When two or more Quantities connected by the Signs + or — are to be divided by any single Quantity, and the Quantities in the Dividend are different from those in the Divisor, then having set down all the Quantities in the Dividend with their Signs and Co-efficients, draw a Line under them all, under which place the Divisor as before, and this will be the Quotient required.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
Divide	$2 a + 3 b$	$7 y - 2 m$	$15 z - 7 d a$
By	$5 m$	$3 n$	$4 y$
Quotient	$\frac{2 a + 3 b}{5 m}$	$\frac{7 y - 2 m}{3 n}$	$\frac{15 z - 7 d a}{4 y}$

Exam. 1. Because $2 a + 3 b$ the Dividend and $5 m$ the Divisor are different, therefore place down $2 a + 3 b$, under which draw a Line, and place the Divisor $5 m$, so is $\frac{2 a + 3 b}{5 m}$ the Quotient required.

Exam. 2. Because $7 y - 2 m$ the Dividend and $3 n$ the Divisor are different, therefore place down $7 y - 2 m$, under which draw a Line, and place the Divisor $3 n$, so is $\frac{7 y - 2 m}{3 n}$ the Quotient required.

Exam. 3. Because $15 z - 7 d a$ the Dividend and $4 y$ the Divisor are different, therefore place down $15 z - 7 d a$ the Dividend, under which draw a Line, and place $4 y$ the Divisor, so is $\frac{15 z - 7 d a}{4 y}$ the Quotient required.

Divide	$4 m a - 3 d$	$7 d b - 5 x z$	$19 m - 15 p$
By	$5 z$	$3 y$	$7 y$
Quotient	$\frac{4 m a - 3 d}{5 z}$	$\frac{7 d b - 5 x z}{3 y}$	$\frac{19 m - 15 p}{7 y}$

 Divide

Divide	$3dz - 5b$	$7ym - 3dn$	$25dp + 5yz - 17m$
By	$2y$	$5u$	$7a$
Quotient	$\frac{3dz - 5b}{2y}$	$\frac{7ym - 3dn}{5u}$	$\frac{25dp + 5yz - 17m}{7a}$

29. If there are two or more Quantities connected by the Signs + or —, to be divided by two or more Quantities connected by the Signs + or —, but the Quantities in the Dividend are different from those in the Divisor, it is only placing down the Dividend as before, under which drawing a Line, and place in like Manner the Divisor, and this will be the Quotient required.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
Divide	$2a + m$	$5y - 7d$	$-14m + 5z - 11x$
By	$5d + 3y$	$3a + 2m$	$3y - 2d$
Quotient	$\frac{2a + m}{5d + 3y}$	$\frac{5y - 7d}{3a + 2m}$	$\frac{-14m + 5z - 11x}{3y - 2d}$

Exam. 1. Because the Quantities in the Dividend and Divisor are unlike, therefore place down $2a + m$ the Dividend with its Co-efficients and Signs, under which draw a Line, and place $5d + 3y$ the Divisor, so is $\frac{2a + m}{5d + 3y}$ the Quotient required.

Exam. 2. Because $5y - 7d$ the Dividend is different from $3a + 2m$ the Divisor, therefore place down $5y - 7d$ the Dividend, under which draw a Line, and place $3a + 2m$ the Divisor, so is $\frac{5y - 7d}{3a + 2m}$ the Quotient required.

Exam. 3. Because $-14m + 5z - 11x$ the Dividend is different from $3y - 2d$ the Divisor, therefore place down $-14m + 5z - 11x$ the Dividend, draw a Line under it, and place $3y - 2d$ the Divisor, and $\frac{-14m + 5z - 11x}{3y - 2d}$ is the Quotient required.

Divide	$4m - 5y$	$-21pm + 19zy$	$14yz - 9dx$
By	$3x + 2z$	$5d - 2b$	$-3m + 5n$
Quotient	$\frac{4m - 5y}{3x + 2z}$	$\frac{-21pm + 19zy}{5d - 2b}$	$\frac{14yz - 9dx}{-3m + 5n}$

Divide

Divide	$-4a+5m-3d$	$4a+3y-5x$	$2a+3y$
By	$7z-8y$	$-7d+11m$	$-5z-7n$
Quotient	$\frac{-4a+5m-3d}{7z-8y}$	$\frac{4a+3y-5x}{-7d+11m}$	$\frac{2a+3y}{-5z-7n}$

30. It may be juſt obſerved to the Learner, that when any Quantity is divided by itſelf, or the Dividend and Diviſor are alike, that then the Quotient will be *Unity*, or 1. And if the Signs of the Quantities to be divided are alike, the Quotient muſt have the Sign +; but if the Signs of the Quantities to be divided are unlike, then to the Quotient, or 1, prefix the Sign —.

Divide	$2ab$	$14mn$	$-5dz$	$+7y$
By	$2ab$	$14mn$	$-5dz$	$-7y$
Quotient	1	1	1	-1

For by Art. 25. if we divide the Co-efficients, the Quotient will be Unity, or 1; then, by Art. 20. rejecting all thoſe Quantities that are alike, both in the Dividend and Diviſor, the Quantities all vaniſh, and there will be none to be joined to the *Unity*, or 1; whence, in ſuch Caſes as theſe, *Unity*, or 1, is the Quotient required.

It may be further obſerved, that if an abſolute or pure Number is the Diviſor, the Co-efficients in the Dividend, if there are more than one, muſt be divided by the Diviſor, and to each of theſe Quotients join the reſpective Quantities of the Dividend, as at Art. 26.

Divide	$24ma+18yz$	$16za+24ym$	$+14yd+35z$
By	6	8	-7
Quotient	$4ma+3yz$	$2za+3ym$	$-2yd-5z$

But if the Diviſor will not exactly divide the Co-efficients of the Dividend, then place the Dividend and Diviſor in the Manner of Vulgar Fractions, as in the foregoing Articles.

The Method of dividing compound Quantities by one another, where the Operation is continued as in common Arithmetic, being generally perplexing to Learners, it will be explained in the Method of ſolving *Quadratic* Equations, this Diviſion not being neceſſary in the preſent Deſign before we come to that Part of the Work.

INVOLUTION.

31. THIS is only raising of Powers from any given Root, and therefore is performed by Multiplication: For the Quantity which is given being multiplied by itself will be the Square of that Quantity, that Product being multiplied by the given Quantity, this Product will be the Cube of that Quantity, and that Product multiplied again by the given Quantity, will be the fourth Power of that Quantity; and so on as in common Arithmetick.

To find the Cube of	a	To find the Cube of	b
	a		b
The Square of a	aa	The Square of b	bb
	a		b
The Cube of a	aaa	The Cube of b	bbb

To find the Cube of —— ——	$2y$
	$2y$
Now $2y$ multiplied by $2y$ the Product will be by Art. 13.	$4yy$ the Square of $2y$
	$2y$
And $4yy$ multiplied by $2y$, the Product will be by Art. 13.	$8yyy$ the Cube of $2y$

To find the Cube of —— ——	$3z$
	$3z$
Now $3z$ multiplied by $3z$, the Product will be by Art. 13.	$9zz$ the Square of $3z$
	$3z$
And $9zz$ multiplied by $3z$, the Product will be by Art. 13.	$27zzz$ the Cube of $3z$

To

To find the 4th Power of $-2x$

$-2x$

Now $-2x$ multiplied by $-2x$, the Product is by Art. 9. and 13. } $4xx$ the Square of $-2x$

$-2x$

And $4xx$ multiplied by $-2x$, the Product is by Art. 13. and 16. } $-8xxx$ the Cube of $-2x$

$-2x$

And $-8xxx$ multiplied by $-2x$, the Product is by Art. 9. and 13. } $16xxxx$ the 4th Power of $-2x$.

In like Manner any other single Quantity may be raised to any required Power, and if in the given Quantity there are more Letters than one, it is done in the same Manner.

To find the 4th Power of $2ab$

$2ab$

$4aabb$ the Square of $2ab$

$2ab$

$8aaabbb$ the Cube of $2ab$

$2ab$

$16aaaabbbb$ the 4th Power of $2ab$.

32. If there are two or more Quantities connected by the Signs $+$ or $-$, to be raised to any given Power, it is still performed by common Multiplication. Two Quantities when connected by the Sign $+$, is commonly called a *Binomial*.

To raise the *Binomial*, or $a+b$ to the third Power or Cube.

$a+b$

$aa+ab$ the Prod. of $a+b$ multip. by a, by Art. 10.

$ab+bb$ the Prod. of $a+b$ multip. by b, by Art. 10.

$aa+2ab+bb$ the Sum of these two Products, or the Square of $a+b$.

$a+b$

$aaa+2aab+abb$ the Product of $aa+2ab+bb$ multiplied by a, by Art. 10.

$aab+2abb+bbb$ the Product of $aa+2ab+bb$ multiplied by b, by Art. 10.

$aaa+3aab+3abb+bbb$ the Sum of these two Products, or the Cube of $a+b$.

When

When two Quantities are connected by the Sign —, it is commonly called a *Residual.*

To raise the *Residual,*

or $x - y$ to the third Power or Cube.

$x - y$

$xx - xy$ the Product of $x - y$ multiplied by x.

$-xy + yy$ the Product of $x - y$ multiplied by $-y$.

$xx - 2xy + yy$ the Sum of these two Products, or the Square of $x - y$.

$x - y$

$xxx - 2xxy + xyy$ the Product of $xx - 2xy + yy$ multiplied by x.

$-xxy + 2xyy - yyy$ the Product of $xx - 2xy + yy$ multiplied by $-y$.

$xxx - 3xxy + 3xyy - yyy$ the Sum of these two Products, or the Cube of $x - y$.

And if these compound Quantities have Co-efficients, the Work still proceeds as at Art. 18.

To raise the *Binomial,*

or $2a + 3b$ to the third Power.

$2a + 3b$

$4aa + 6ab$ the Product of $2a + 3b$ multiplied by $2a$.

$6ab + 9bb$ the Product of $2a + 3b$ multiplied by $3b$.

$4aa + 12ab + 9bb$ the Sum of these two Products, or the Square of $2a + 3b$.

$2a + 3b$

$8aaa + 24aab + 18abb$ the Product of $4aa + 12ab + 9bb$ multiplied by $2a$.

$12aab + 36abb + 27bbb$ the Product of $4aa + 12ab + 9bb$ multiplied by $3b$.

$8aaa + 36aab + 54abb + 27bbb$ the Sum of these two Products, or the Cube of $2a + 3b$.

To

To raise $3m+2y$ to the third Power.

$$\begin{array}{l}
3m+2y\\
\hline
9mm+6my\\
6my+4yy\\
\hline
9mm+12my+4yy \text{ the Square of } 3m+2y\\
3m+2y\\
\hline
27mmm+36mmy+12myy\\
18mmy+24myy+8yyy\\
\hline
27mmm+54mmy+36myy+8yyy \text{ the Cube of } 3m+2y.
\end{array}$$

To raise $a-2b$ to the third Power.

$$\begin{array}{l}
a-2b\\
\hline
aa-2ab\\
\quad -2ab+4bb\\
\hline
aa-4ab+4bb \text{ the Square of } a-2b\\
a-2b\\
\hline
aaa-4aab+4abb\\
\quad -2aab+8abb-8bbb\\
\hline
aaa-6aab+12abb-8bbb \text{ the Cube of } a-2b.
\end{array}$$

In this Example I have placed the same Quantities under each other, for the more commodious adding them, though this is not necessary, and is a Knowledge the Learner will acquire from his own Observation.

EVOLUTION.

33. THIS is the Extraction of Roots, and therefore opposite to Involution, and as Equations in which the unknown Quantity rises above the Square are generally affected, and resolved by the Method of *Converging Series*, we shall consider the Square Root only; and give such Directions that the Learner may generally know, whether the Square Root of such Quantities as commonly occur in the Solution of Questions can be extracted or not.

Now so many Times as any Letter is repeated, so high is the Power of that Letter said to be. Thus, a is a to the first Power: aa is a to the second Power or Square: and $aaaa$ is a to the fourth Power, &c. as in Involution.

And to extract the Root of any simple Quantity, consider how many Times the Letter is repeated, or how high the Power of it is, and if it appears to be the second, third, fourth, or any other Power, divide that Figure which expresses the Heighth of the Power by 2, and if it does not divide it exactly, it is a *Surd* Quantity, and has no Square Root; but if it divides it exactly, set down the Quantity whose Root you are extracting as many Times as the Quotient of the above Division directs, and that will be the Square Root required.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
To extract the Square Root of	aa	$bbbb$	$bbbbbb$
The Square Root is	a	bb	bbb

Exam 1. Here a is repeated twice, or to the second Power; now dividing 2 by 2 the Quotient is 1, therefore setting down a once, or a, it is the Square Root required.

Exam. 2. Here b is repeated four times, or to the fourth Power; now dividing 4 by 2 the Quotient is 2, therefore setting down b twice, or bb, it is the Square Root required.

Exam. 3. Here b is repeated six times, or to the sixth Power; now dividing 6 by 2 the Quotient is 3, therefore setting down b three times, or bbb, it is the Square Root required.

The Truth of these Operations are proved by Multiplication, for if the Work is right, the Square Root being multiplied by itself will produce the Quantity from which the Root was extracted. Thus in Example 2,

The Square Root is	bb
Which being multiplied by itself	bb
The Product is the given Square	$bbbb$

And so of any other Example.

	Exam. 4.	*Exam.* 5.
To extract the Square Root of	$aaaa$	$dddddd$
The Square Root is	aa	ddd

Exam. 4. Here a is repeated four times, or to the fourth Power; now dividing 4 by 2 the Quotient is 2, which shows that a must be repeated twice, that is, aa is the Square Root required.

Exam.

Exam. 5. Here d is repeated ſix times, or to the ſixth Power; now dividing 6 by 2 the Quotient is 3, which ſhows that d muſt be repeated three times, and conſequently ddd is the Square Root required.

And if the Quantity, whoſe Root is to be extracted, has different Letters, then conſider if the Number of Times each Letter is repeated can be divided by 2 without any Remainder, and if they can, ſet down each Letter ſo many Times as the Quotient of the reſpective Diviſion directs, and joining them, this will be the Square Root required; but if the Number of Times any one Letter is repeated cannot be divided by 2, then the whole Quantity has no Square Root.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
To extract the Square Root of	$aabbbb$	$aaaadddd$	$mmpp$
The Square Root is	abb	$aadd$	mp

Exam. 1. Here a is repeated twice, and 2 being divided by 2 the Quotient is 1, which ſhows a muſt be taken only once, or a. Now b is repeated four Times, or to the fourth Power, and 4 being divided by 2 the Quotient is 2, which ſhows b muſt be repeated twice, or bb, now joining a to bb, and abb is the Square Root required.

Exam. 2. Here a is repeated to the fourth Power, and dividing 4 by 2 the Quotient is 2, which ſhows that a muſt be repeated twice, that is, it muſt be aa: Again, d is repeated to the fourth Power, and dividing 4 by 2, the Quotient is 2, which ſhows d muſt be repeated to the ſecond Power, or dd. Now joining aa to dd, we have $aadd$ for the Square Root required.

By the ſame Method of reaſoning we ſhall find in Example 3, that the Square Root of $mmpp$ is mp.

But when it is found that the given Quantity has not ſuch a Root as is required, then the Square Root of it is expreſſed by prefixing this Sign $\surd$ before it.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
Required the Square Root of	a	bbb	$ddddd$
The Square Root is	$\surd\, a$	$\surd\, bbb$	$\surd\, ddddd$.

Exam. 1. Becauſe a is only repeated once, and as we cannot divide 1 by 2, and have the Quotient a whole Number, therefore I conclude a is a *Surd* Quantity, and accordingly, to expreſs the

Square Root of a, prefix the Sign $\surd$ to it, so is $\surd a$ the Square Root required.

Exam. 2. Here b is repeated three times, and because 3 cannot be divided by 2, and have no Remainder, therefore I conclude bbb is a *Surd* Quantity, and to express the Square Root of it, prefix the Sign $\surd$ to it, so is $\surd bbb$ the Square Root required.

Exam. 3. Here d being repeated five times, and as we cannot divide 5 by 2, and have no Remainder, therefore I conclude that $ddddd$ is a *Surd* Quantity, and to express the Square Root of it, prefix the Sign $\surd$ to it, so is $\surd ddddd$ the Square Root required.

34. But to extract the Square Root of compound Quantities, or those connected by the Signs $+$ or $-$, observe,

First, There must be three Quantities to make it a Square, for $a+b$ multiplied by itself, or squared, the Product is $aa+2ab+bb$, by Art. 32. whence if there are only two Quantities it is a *Surd.* I take no Notice of any greater Number of Quantities than three, which may compose a Square, as they seldom occur in any Operation.

Secondly, Whether these three Quantities have two different Letters only; there may be Cases in which there are more than two different Letters in these three Quantities, but as they seldom happen, I choose not to perplex the Learner with them.

Thirdly, If two of these three Quantities are pure Powers of those two Letters; that is, in the Square of $a+b$ there is aa and bb, pure Powers of the Quantities a and b.

Fourthly, Whether both these pure Powers of the two different Letters have the Sign $+$ before them.

Fifthly, If the third of the above three Quantities is twice the Product of the Square Root of the two pure Powers of the two different Letters, that is, the Square of $a+b$ being $aa+2ab+bb$, the Quantity $2ab$ is twice the Product of the Square Root of aa and bb; and this Quantity may have either the Sign $+$ or $-$.

Now if the given Quantity, whose Root is to be extracted, answers these Particulars, its Square Root may be extracted thus.

Sixthly, Extract the Square Root of the two pure Powers of the two different Letters, according to the Directions at Art. 33.

Seventhly, If the Quantity mentioned at the *fifth* Particular has the *negative* Sign, connect the two Roots mentioned in the last Particular with the Sign $-$, and it will be the Square Root required.

Eighthly,

Eighthly, But if the Quantity mentioned at the *fifth* Particular has the Sign +, then connect those two Roots with the Sign +, and this will be the Square Root required.

Now let it be required to extract the Square Root of $aa + 2ab + bb$.

Here are three Quantities by the *first* Particular.

They have likewise two different Letters, *viz.* a and b, by the *second* Particular.

Two of these Quantities, *viz.* aa and bb, are pure Powers of the two Letters a and b, by the *third* Particular.

And both these pure Powers, *viz.* aa and bb, have the Sign +, by the *fourth* Particular.

Now suppose we neglect the Consideration of the *fifth* Particular, and attempt the Extraction of the Root by the *sixth* Particular.

Then the Square Root of aa, is by Art. 33. —— a

And the Square Root of bb, is by the same —— b

And now the third Quantity $2ab$ being twice the Product of the Roots a and b, and having the Sign +,

Therefore by the *eighth* Particular, I connect a and b with the Sign +, then it is —— —— $a+b$

Hence I suppose $a+b$ to be the Square Root of $aa+2ab+bb$.

But to prove the Truth of the Operation, multiply the Root by itself, and if the Product agrees with the given Quantity, in its *Quantities*, *Signs*, and *Co-efficients*, the Work is right; if not the Work is either erroneous, or has no Square Root, and is a *Surd* Quantity.

The Root of the last Example was supposed to be	$a+b$
Which multiplied by itself	$a+b$
	$aa+ab$
	$ab+bb$
	$aa+2ab+bb$

The Product is the given Quantity, which proves that $a+b$ is the Square Root of $aa+2ab+bb$.

Required the Square Root of $aa+2za+zz$.

Here are three Quantities by the *first* Particular.

They have likewise two different Letters, a and z, by the *second* Particular.

Two of these Quantities, *viz.* aa and zz, are pure Powers of a and z, by the *third* Particular, whose Square Roots are a and z.

And both these Powers have the Sign + by the *fourth* Particular.

Now the third Quantity $2za$ is twice the Product of a and z, the Square Roots of the two pure Powers aa and zz.

Then to extract the Square Root of $aa + 2za + zz$ by the *sixth* Particular.

The Square Root of aa by Art. 33. is ——— a

The Square Root of zz by the same is ——— z

Because the third Quantity $2az$ has the Sign $+$, therefore by the *eighth* Particular, connect a and z with the Sign $+$, and $a + z$ is the Square Root required.

To try if the Square Root is $a + z$
Multiply it by itself $a + z$

$$\begin{array}{l} a + z \\ a + z \\ \hline aa + az \\ \quad\; az + zz \\ \hline aa + 2az + zz \end{array}$$

The Product $aa + 2az + zz$, agreeing with the given Quantity, in the Quantities, Signs, and Co-efficients, it appears that $a + z$ is the Square Root required.

To extract the Square root of $mm - 2mp + pp$.

Here are three Quantities by the *first* Particular.

They have likewise two different Letters m and p, by the *second* Particular.

Two of these three Quantities, *viz.* mm and pp are pure Powers of m and p, by the *third* Particular.

And both these Powers have the Sign $+$, by the *fourth* Particular.

Likewise the third Quantity $-2mp$ is twice the Product of m and p, the Square Roots of the two pure Powers mm and pp.

Then according to the *sixth* Particular, the Square Root of mm is ——— ——— ——— m

By the same, the Square Root of pp is ——— ——— p

But as the third Quantity $2mp$ has the Sign $-$, therefore by the *seventh* Particular connect m and p with the Sign $-$, and $m - p$ is the Square Root required.

To try if the Square Root is $m - p$
Multiply it by itself $m - p$

$$\begin{array}{l} m - p \\ m - p \\ \hline mm - mp \\ \quad\;\; - mp + pp \\ \hline mm - 2mp + pp \end{array}$$

The Product $mm - 2mp + pp$, agreeing in every thing with the given Quantity, it proves $m - p$ is the Square Root required.

By

By the same Method of reasoning it will be found that the Square Root of $zz + 2zx + xx$, is $z + x$.

And that the Square Root of $aa - 2ad + dd$, is $a - d$.

And that the Square Root of $xx - 2xm + mm$, is $x - m$.

And if it was required to extract the Square Root of $aa + ba + \frac{bb}{4}$, in extracting the Root of the Fractional Quantity extract the Root of the Numerator for a new Numerator, and the Root of the Denominator for a new Denominator.

Here the two pure Powers are aa and $\frac{bb}{4}$.

But the Square Root of aa is — — a

And the Square Root of $\frac{bb}{4}$ is — — $\frac{b}{2}$

And connecting these we have — — $a + \frac{b}{2}$

The Square Root required.

To prove the Truth of this Operation, multiply $a + \frac{b}{2}$ by itself,

$$\begin{array}{r} a + \frac{b}{2} \\ a + \frac{b}{2} \\ \hline aa + \frac{ab}{2} \qquad\quad \\ \frac{ab}{2} + \frac{bb}{4} \\ \hline aa + ab + \frac{bb}{4} \end{array}$$

a multiplied by a the Product is aa, and $\frac{b}{2}$ multiplied by a is $\frac{ab}{2}$, (for making a an improper Fraction $\frac{a}{1}$ as in common Arithmetic, and multiplying the two Numerators a and b for a new Numerator, and the two Denominators 1 and 2 for a new Denominator, we have $\frac{ab}{2}$) and $\frac{b}{2}$ multiplied by $\frac{b}{2}$ produces $\frac{bb}{4}$ by the same Rule; and in the Products the Fractions $\frac{ab}{2}$ and $\frac{ab}{2}$ having the same Denominator, adding them according-

ing

ing to the Rule for Addition of Vulgar Fractions in Arithmetic, the Sum is $\frac{2ab}{2}$, but rejecting the 2 by the Rule for Division of *Algebra* the Sum is ab.

Therefore when any one of the Quantities appears in a *Fractional* Manner, we must extract the Square Root of both the Numerator and Denominator, placing the Square Root of the Numerator for a new Numerator, and the Square Root of the Denominator for a new Denominator, and try the Work as before.

But if we cannot extract the Square Root of both the Numerator and Denominator, then we conclude the given Quantity to be a Surd.

Now by this Reasoning we shall find the Square Root of $xx + xa + \frac{aa}{4}$, to be $x + \frac{a}{2}$.

And that the Square Root of $mm - my + \frac{yy}{4}$, is $m - \frac{y}{2}$.

Suppose it was required to extract the Square Root of $xx + 2xn - nn$.

Here are three Quantities by the *first* Particular.

They have likewise two different Letters, x and n, by the *second* Particular.

Two of these three Quantities, *viz.* xx and nn, are pure Powers of x and n.

But both these Powers have not the Sign $+$, for it is $-nn$, therefore by the *fourth* Particular, I conclude that the given Quantity $xx + 2xn - nn$ is a *Surd* Quantity, and its Square Root cannot be extracted any otherwise than by prefixing the Sign $\surd$ to it, as in Art. 33. Thus, $\sqrt{xx + 2xn - nn}$ is, or expresses the Square Root of $xx + 2xn - nn$.

Let it be required to extract the Square Root of $aa + 5ab + bb$.

Here are three given Quantities by the *first* Particular.

They have likewise two different Letters, a and b, by the *second* Particular.

Two of these Quantities, *viz.* aa and bb, are pure Powers of a and b.

And both these Powers have the Sign $+$ by the *fourth* Particular.

But then the third Quantity $5ab$ is not twice the Product of the Square Roots of aa and bb; for their Roots being a and b, if they are multiplied the Product is ab, and that being multiplied by 2 it is $2ab$: Whereas the third Quantity in the given Example

Example is $5ab$. Hence, I conclude that $aa + 5ab + bb$ is a *Surd* Quantity, and to express its Square Root I prefix to it the Sign $\surd$, so will $\sqrt{aa + 5ab + bb}$ be the Square Root of $aa + 5ab + bb$.

And if it was required to extract the Square Root of $aa + 2ab + \frac{bb}{5}$, it will be found a *Surd* Quantity, it being impossible to extract the Square Root of 5, therefore prefix the Sign $\surd$ to $aa + 2ab + \frac{bb}{5}$, and then $\sqrt{aa + 2ab + \frac{bb}{5}}$ is the Square Root required.

For the same Reason the Square Root of $xx + 2xa + \frac{aa}{3}$ is $\sqrt{xx + 2xa + \frac{aa}{3}}$, it being impossible to extract the Square Root of 3.

When the Radical Sign or $\surd$ is to be prefixt to the Whole of any compound Quantity, draw the Top of the Sign over all those Quantities, which shews that they are all included under that Sign; for if the Sign was not to be drawn over all of them, it may be thought the Square Root of that Quantity was only to be extracted which stands next the Radical Sign.

To extract the Square Root of $aaaa + 2aab + bb$.

Here are three given Quantities by the *first* Particular.

They have likewise two different Letters, a and b, by the *second* Particular.

Two of these Quantities, *viz.* $aaaa$ and bb, are pure Powers of a and b, by the *third* Particular.

And both these Powers have the Sign $+$, by the *fourth* Particular.

And the third Quantity $2aab$ is twice the Product of the Square Roots of $aaaa$ and bb.

For by the *sixth* Particular, the Square Root of $aaaa$ is aa

And by the same, the Square Root of bb is — — b

And as the third Term $2aab$ in the given Quantity has the Sign $+$, by the *eighth* Particular connect aa and b, the two Roots of $aaaa$ and bb, with the Sign $+$, so is $aa + b$ the Square Root of $aaaa + 2aab + bb$.

To prove which put down the supposed Square Root	$aa+b$
Which multiplied by itself	$aa+b$
	$aaaa+aab$
	$aab+bb$
	$aaaa+2aab+bb$

The Product $aaaa+2aab+bb$, agreeing with the given Quantity in every Particular, proves the Square Root to be as above.

To extract the Square Root of $yyyy-2yyx+xx$.

Here the given Quantities agree with the first five Particulars as before.

By the *sixth* Particular I find the Square Root of $yyyy$ is yy

By the same, that the Square Root of xx is — — x

But as the third Term $-2yyx$ in the given Quantity has the Sign —, therefore by the *seventh* Particular I connect yy and x the two Roots with the Sign —, and say, or suppose $yy-x$ to be the Square Root required.

To prove which put down the supposed Root	$yy-x$
Which multiply by itself	$yy-x$
	$yyyy-yyx$
	$-yyx+xx$
The Product, agreeing with the given Quantity	$yyyy-2yyx+xx$

By the same Method we shall find the Square Root of $nnnn+2nnd+dd$ to be $nn+d$.

And that the Square Root of $xxxx+2xxyy+yyyy$ is $xx+yy$.

And we shall find that $dddd+3ddy+yy$ is a *Surd* Quantity, and its Square Root must be expressed by prefixing the Radical Sign to it, thus $\sqrt{dddd+3ddy+yy}$.

We shall likewise find that $-pppp+2ppy+yy$ is a *Surd* Quantity, and to extract its Square Root, is only to prefix to it the Radical Sign, thus $\sqrt{-pppp+2ppy+yy}$.

Of

Of Surd Quantities.

THESE are such Quantities whose Roots cannot be exactly extracted, and as they arise in the Resolution of *Algebraic* Questions, we shall explain so much of them only, as is necessary to the present Design.

Addition of Surd Quantities, in which there are two Cases.

35. *Case* 1. When the Quantities under the Radical Signs are alike, add the rational Quantities, or those which are without the Radical Sign together, by the Rules of Addition at *Art.* 1, 2, 3, 4, 5, 6, and to this join the *Surd* Quantities, and this will be the Sum required.

And if there appears to be no rational Quantities without the Radical Sign, then *Unity*, or 1, is always supposed to be the rational Quantity.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
To	$\sqrt{am}$	$2\sqrt{dy}$	$6m\sqrt{d+a}$	$5y\sqrt{dm+z}$
Add	$\sqrt{am}$	$\sqrt{dy}$	$4m\sqrt{d+a}$	$y\sqrt{dm+z}$
Sum	$2\sqrt{am}$	$3\sqrt{dy}$	$10m\sqrt{d+a}$	$6y\sqrt{dm+z}$

Exam. 1. There being no rational Quantities, therefore Unity, or 1, is the rational Quantity to each. Now 1 added to 1 makes 2, to which joining the Surd $\sqrt{am}$, we have $2\sqrt{am}$, the Sum required.

Exam. 2. The rational Quantities being 2 and 1, their Sum is 3, to which joining $\sqrt{dy}$ we have $3\sqrt{dy}$, the Sum required.

Exam. 3. The rational Quantities are $6m$ and $4m$, which being added make $10m$, to which joining the Surd $\sqrt{d+a}$ we have $10m\sqrt{dm+a}$, the Sum required.

Exam. 4. The rational Quantities are $5y$ and y, which being added make $6y$, to which joining the Surd $\sqrt{dm+z}$ we have $6y\sqrt{dm+z}$, the Sum required.

	Exam. 5.	*Exam.* 6.	*Exam.* 7.
To	$13yd\sqrt{z-x}$	$15z\sqrt{da+p}$	$-7m\sqrt{da-y}$
Add	$5yd\sqrt{z-x}$	$-3z\sqrt{da+p}$	$-2m\sqrt{da-y}$
Sum	$18yd\sqrt{z-x}$	$12z\sqrt{da+p}$	$-9m\sqrt{da-y}$

Exam. 5. The rational Quantities $13yd$ and $5yd$ being added make $18yd$, to which joining the Surd Quantity $\sqrt{z-x}$ we have $18yd\sqrt{z-x}$, the Sum required.

 Exam.

Exam. 6. The rational Quantities $15z$ and $-3z$ being added, their Sum by *Art.* 3. is $12z$, to which joining the Surd $\sqrt{da+p}$ we have $12z\sqrt{da+p}$, the Sum required.

Exam. 7. The rational Quantities $-7m$ and $-2m$ being added make $-9m$, to which joining $\sqrt{da-y}$, we have $-9m\sqrt{da-y}$, the Sum required.

To	$-2y\sqrt{ma+m}$	$-15m\sqrt{da-zp}$	$16dp\sqrt{14+p}$
Add	$-3y\sqrt{ma+m}$	$7m\sqrt{da-zp}$	$-12dp\sqrt{14+p}$
Sum	$-5y\sqrt{ma+m}$	$-8m\sqrt{da-zp}$	$4dp\sqrt{14+p}$
To	$4y\sqrt{zd-za}$	$5y\sqrt{mp+x}$	$7z\sqrt{ma-d}$
Add	$3y\sqrt{zd-za}$	$-4y\sqrt{mp+x}$	$-8z\sqrt{ma-d}$
Sum	$7y\sqrt{zd-za}$	$y\sqrt{mp+x}$	$-z\sqrt{ma-d}$

36. *Case* 2. When the Letters under the *Radical* Sign are *different*, then place them down one after the other with the ſame Signs they have in the Queſtion, in the Manner as at Art. 6. and this will be the Sum required.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
To	$\sqrt{a}$	$\sqrt{b+m}$	$m\sqrt{da+y}$
Add	$\sqrt{b}$	$\sqrt{d+y}$	$m\sqrt{z}$
Sum	$\sqrt{a}+\sqrt{b}$	$\sqrt{b+m}:+\sqrt{d+y}$	$m\sqrt{da+y}:+m\sqrt{z}$

Exam. 1. The Letters under the radical Signs being different put down $\sqrt{a}$, then becauſe $\sqrt{b}$ has the Sign $+$, therefore after $\sqrt{a}$ put $+$, after which put $\sqrt{b}$, and $\sqrt{a}+\sqrt{b}$, is the Sum required.

Exam. 2. The Letters under the radical Signs being different put down $\sqrt{b+m}$: after which place two Dots to ſhow that *Surd* goes no farther, then becauſe $\sqrt{d+y}$ has the Sign $+$, therefore after the Quantity $\sqrt{b+m}$: put $+$, and after that the Surd $\sqrt{d+y}$, and we have $\sqrt{b+m}:+\sqrt{d+y}$, the Sum required.

Exam. 3. The Letters under the radical Signs being different put down $m\sqrt{da+y}$: and becauſe the Quantity $m\sqrt{z}$ has the Sign $+$, therefore after $m\sqrt{da+y}$: put the Sign $+$, after which put the Quantity $m\sqrt{z}$, and we have $m\sqrt{da+y}:+m\sqrt{z}$, the Sum required.

Exam.

Exam. 4.

To $y\sqrt{da}$
Add $-z\sqrt{m}$

Sum $y\sqrt{da}-z\sqrt{m}$

Exam. 5.

$-5\sqrt{da-y}$
$-2m\sqrt{zm}$

$-5\sqrt{da-y}:-2m\sqrt{zm}$

Exam. 6.

$-2m\sqrt{bz+n}$
$3y\sqrt{dz-b}$

$-2m\sqrt{bz+n}:+3y\sqrt{dz-b}$

Exam. 4. The Letters under the radical Signs being different put down $y\sqrt{da}$, and becauſe $-z\sqrt{m}$ has the Sign —, therefore after $y\sqrt{da}$ put the Sign —, and after that the Quantity $z\sqrt{m}$, and $y\sqrt{da}-z\sqrt{m}$ is the Sum required.

Exam. 5. The Letters under the radical Signs being different put down $-5\sqrt{da-y}$: and becauſe $-2m\sqrt{zm}$ has the Sign —, therefore after $-5\sqrt{da-y}$: put the Sign —, and after that the Quantity $2m\sqrt{zm}$, and $-5\sqrt{da-y}$: $-2m\sqrt{zm}$ is the Sum required.

Exam. 6. Becauſe the Letters under the radical Signs are different I put down $-2m\sqrt{bz+n}$, but $3y\sqrt{dz-b}$ having the Sign +, therefore after $-2m\sqrt{bz+n}$: put the Sign +, and after that the Quantity $3y\sqrt{dz-b}$: and $-2m\sqrt{bz+n}:+3y\sqrt{dz-b}$ is the Sum required.

To $-5\sqrt{da}$
Add $7\sqrt{m}$

Sum $-5\sqrt{da}+7\sqrt{m}$

$m\sqrt{bma}$
$3\sqrt{yp+q}$

$m\sqrt{bma}+3\sqrt{yp+q}$

To $-3y\sqrt{p+r}$
Add $m\sqrt{d}$

Sum $-3y\sqrt{p+r}:+m\sqrt{d}$

$14m\sqrt{da+pz}$
$7\sqrt{z+p}$

$14m\sqrt{da+pz}:+7\sqrt{z+p}$

To $-5y\sqrt{dp-z}$
Add $7y\sqrt{zm+a}$

Sum $-5y\sqrt{dp-z}:+7y\sqrt{zm+a}$

Subſtraction of Surd Quantities, in which there are two Caſes.

37. *Caſe* 1. When the Letters under the *Radical* Signs are *alike*, ſubſtract the rational Quantities from the rational Quantities by Art. 7. and to the Difference join the Surd Quantities, which will be the Remainder required.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
From	$5\sqrt{da}$	$5m\sqrt{mz}$	$14y\sqrt{d+z}$	$21pm\sqrt{db-r}$
Subſtract	$3\sqrt{da}$	$2m\sqrt{mz}$	$3y\sqrt{d+z}$	$19pm\sqrt{db-r}$
Remains	$2\sqrt{da}$	$3m\sqrt{mz}$	$11y\sqrt{d+z}$	$2pm\sqrt{db-r}$

Exam. 1. The rational Quantities are 5 and 3, ſubſtracting 3 from 5 there remains 2, to which joining the Surd $\sqrt{da}$ we have $2\sqrt{da}$, the Remainder required.

Exam. 2. The rational Quantities are $5m$ and $2m$, ſubſtracting $2m$ from $5m$ there remains $3m$, to which joining the Surd $\sqrt{mz}$ we have $3m\sqrt{mz}$, the Remainder required.

Exam. 3. The rational Quantities are $14y$ and $3y$, ſubſtracting $3y$ from $14y$ there remains $11y$, to which joining the Surd $\sqrt{d+z}$ we have $11y\sqrt{d+z}$, the Remainder required.

Exam. 4. The rational Quantities are $21pm$, and $19pm$, ſubſtracting $19pm$ from $21pm$, there remains $2pm$, to which joining the Surd $\sqrt{db-r}$ we have $2pm\sqrt{db-r}$, the Remainder required.

	Exam. 5.	*Exam.* 6.	*Exam.* 7.
From	$17d\sqrt{ba}$	$-5y\sqrt{d+a}$	$-5m\sqrt{d+ab}$
Subſtract	$-4d\sqrt{ba}$	$3y\sqrt{d+a}$	$-6m\sqrt{d+ab}$
Remains	$21d\sqrt{ba}$	$-8y\sqrt{d+a}$	$m\sqrt{d+ab}$

Exam. 5. The rational Quantities are $17d$ and $-4d$: Now to ſubſtract $-4d$ from $17d$, by the Rule for Subſtraction at Art. 7. change the Sign of $-4d$, or ſuppoſe it to be changed, then $-4d$ becomes $+4d$ or $4d$; then by Art. 7. if we add $17d$ to $4d$ it is $21d$, which is the Remainder that ariſes by ſubſtracting $-4d$ from $17d$; now to this $21d$ join the Surd $\sqrt{ba}$, and $21d\sqrt{ba}$ is the Remainder required.

Exam. 6. To ſubſtract the rational Quantity $3y$ from $-5y$, we muſt by Art. 7. change or ſuppoſe the Sign of $3y$

to

to be changed, which will make it $-3y$: then by the fame Art. $-3y$ added to $-5y$, it is $-8y$, which is the Remainder that arifes from fubftracting the rational Quantities, therefore to this $-8y$ join the Surd Quantity $\sqrt{d+a}$, and $-8y\sqrt{d+a}$ is the Remainder required.

Exam. 7. Here the rational Quantities are $-5m$ and $-6m$, and by the Rule for Subftraction Art. 7. if we fuppofe the Sign of $-6m$ to be changed, it becomes $+6m$ or $6m$, and then adding $-5m$ to $6m$ it is m, the Remainder arifing from fubftracting the rational Quantities; and if to this m we join the Surd $\sqrt{d+ab}$ we have $m\sqrt{d+ab}$, the Remainder required.

	Exam. 8.	*Exam.* 9.	*Exam.* 10.
From	$21m\sqrt{d+a}$	$-9d\sqrt{mn+p}$	$12y\sqrt{d-an}$
Subftract	$9m\sqrt{d+a}$	$-2d\sqrt{mn+p}$	$-3y\sqrt{d-an}$
Remains	$12m\sqrt{d+a}$	$-7d\sqrt{mn+p}$	$15y\sqrt{d-an}$

	Exam. 11.	*Exam.* 12.	*Exam.* 11.
From	$-4a\sqrt{m-p}$	$14p\sqrt{d-y}$	$-5a\sqrt{x+y}$
Subftract	$2a\sqrt{m-p}$	$-3p\sqrt{d-y}$	$3a\sqrt{x+y}$
Remains	$-6a\sqrt{m-p}$	$17p\sqrt{d-y}$	$-8a\sqrt{x+y}$

	Exam. 14.	*Exam.* 15.	*Exam.* 16.
From	$7a\sqrt{ap}$	$-21\sqrt{ap-ax}$	$-14\sqrt{da-z}$
Subftract	$2a\sqrt{ap}$	$-9\sqrt{ap-ax}$	$7\sqrt{da-z}$
Remains	$5a\sqrt{ap}$	$-12\sqrt{ap-ax}$	$-21\sqrt{da-z}$

The Truth of thefe Operations are proved as in Subftraction of common Numbers. Thus at Example 1. the Remainder is $2\sqrt{da}$, and the Quantity fubftracted was $3\sqrt{da}$, now if we add thefe together by Art. 35. the Sum is $5\sqrt{da}$, which being the fame Quantity from which $3\sqrt{da}$ was fubftracted, it proves the Work to be true.

Again, at Example 6. the Remainder is $-8y\sqrt{d+a}$: the Quantity fubftracted was $3y\sqrt{d+a}$: Now by Art. 35. if to $-8y\sqrt{d+a}$ we add $3y\sqrt{d+a}$, the Sum is $-5y\sqrt{d+a}$, which being the Quantity from which $3y\sqrt{d+a}$ was fubftracted, it proves the Work to be true.

38. *Case* 3. When the Letters under the radical Signs are different, set them down one after the other, as at Art. 36. but in setting them down take Care to change the Signs of those Quantities that are to be substracted, by Art. 7. and this will be the Remainder required.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
From	$2\sqrt{da}$	$2m\sqrt{dp}$	$5y\sqrt{a}$
Substract	$3\sqrt{m}$	$y\sqrt{z}$	$-3\sqrt{b}$
Remains	$2\sqrt{da}-3\sqrt{m}$	$2m\sqrt{dp}-y\sqrt{z}$	$5y\sqrt{a}+3\sqrt{b}$

Exam. 1. The Letters under the radical Signs being different place down $2\sqrt{da}$, and because $3\sqrt{m}$ the Quantity to be substracted has the Sign $+$, therefore after $2\sqrt{da}$ place the Sign $-$, and after that the Quantity $3\sqrt{m}$, and $2\sqrt{da}-3\sqrt{m}$ is the Remainder required.

Exam. 2. Because the Letters under the radical Signs are different put down $2m\sqrt{dp}$, and because $y\sqrt{z}$ the Quantity to be substracted has the Sign $+$, therefore after $2m\sqrt{dp}$ put the Sign $-$, and after that $y\sqrt{z}$, and $2m\sqrt{dp}-y\sqrt{z}$ is the Remainder required.

Exam. 3. Because the Letters under the radical Signs are different put down $5y\sqrt{a}$, but as $-3\sqrt{b}$ the Quantity to be substracted has the Sign $-$, therefore after $5y\sqrt{a}$ put the Sign $+$, and after that $3\sqrt{b}$, and $5y\sqrt{a}+3\sqrt{b}$ is the Remainder required.

	Exam. 4.	*Exam.* 5.
From	$m\sqrt{da+p}$	$-5y\sqrt{a}$
Substract	$2\sqrt{a}$	$-d\sqrt{b}$
Remains	$m\sqrt{da+p}:-2\sqrt{a}$	$-5y\sqrt{a}+d\sqrt{b}$

	Exam. 6.
From	$5m\sqrt{a}$
Substract	$-z\sqrt{p+q}$
Remains	$5m\sqrt{a}+z\sqrt{p+q}$

Exam. 4. Because the Letters under the radical Signs are different put down $m\sqrt{da+p}$, but as $2\sqrt{a}$ the Quantity to be substracted has the Sign $+$, therefore after $m\sqrt{da+p}$ put the Sign $-$, and after that $2\sqrt{a}$, and $m\sqrt{da+p}:-2\sqrt{a}$ is the Remainder required.

Exam.

Exam. 5. Becauſe the Letters under the radical Signs are different put down $-5y\sqrt{a}$, but as $-d\sqrt{b}$ the Quantity to be ſubſtracted has the Sign $-$, therefore after $-5y\sqrt{a}$ put the Sign $+$, and after that $d\sqrt{b}$, and $-5y\sqrt{a}+d\sqrt{b}$ is the Remainder required.

Exam. 6. Becauſe the Letters under the radical Signs are different put down $5m\sqrt{a}$, but as $-z\sqrt{p+q}$ has the Sign $-$ before it, therefore after $5m\sqrt{a}$ put the Sign $+$, and after that $z\sqrt{p+q}$, and $5m\sqrt{a}+z\sqrt{p+q}$ is the Remainder required.

From	$5\sqrt{a+p}$	$m\sqrt{p}$
Subſtract	$m\sqrt{y}$	$-y\sqrt{da-p}$
Remains	$5\sqrt{a+p}:-m\sqrt{y}$	$m\sqrt{p}+y\sqrt{da-p}$
From	$3u\sqrt{d+p}$	$-5n\sqrt{da}$
Subſtract	$2n\sqrt{z-y}$	$-3\sqrt{m}$
Remains	$3u\sqrt{d+p}:-2n\sqrt{z-y}$	$-5n\sqrt{da}+3\sqrt{m}$
From	$-5\sqrt{p+z}$	$14\sqrt{da}$
Subſtract	$3n\sqrt{m}$	$7\sqrt{p+y}$
Remains	$-5\sqrt{p+z}:-3n\sqrt{m}$	$14\sqrt{da}:-7\sqrt{p+y}$

The Truth of theſe Operations are proved in the ſame Manner as in the laſt Article, by adding the Remainder to the Quantity that was ſubſtracted; and if their Sum makes the Quantity from which the other was taken, the Work is true, if not, there is a Miſtake.

Thus at *Example* 1. the Remainder is $2\sqrt{da}-3\sqrt{m}$
To which if we add the Quantity ſubſtracted $3\sqrt{m}$

The Sum is $2\sqrt{da}$, the ſame in the given Example. $2\sqrt{da}$ For in this Addition, adding $+3\sqrt{m}$ to $-3\sqrt{m}$, the Co-efficients and Quantities being the ſame and the Signs contrary, they deſtroy one another, or the Sum is *nothing*, by Art. 5.

Again at *Example* 5. the Remainder is $-5y\sqrt{a}+d\sqrt{b}$
To which if we add the Quantity ſubſtracted $-d\sqrt{b}$

The Sum is $-5y\sqrt{a}$, the ſame as in the given Example. $-5\sqrt{a}$ For here $-d\sqrt{b}$ being added to $d\sqrt{b}$ or $+d\sqrt{b}$, they deſtroy one another

another as in the last Instance. In like Manner the Reader may prove any of the other Examples.

Multiplication of Surd Quantities, in which there are two Cases.

39. *Case* 1. When there are no rational Quantities but Unity joined to the Surd Quantities, then multiply the Surd Quantities, as in Multiplication of rational Quantities, but to their Product prefix the radical Sign.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Multiply	$\sqrt{a}$	$\sqrt{mn}$	$\sqrt{py}$	$\sqrt{zx}$
By	$\sqrt{m}$	$\sqrt{d}$	$\sqrt{z}$	$\sqrt{a}$
Product	$\sqrt{am}$	$\sqrt{mnd}$	$\sqrt{pyz}$	$\sqrt{zxa}$

Exam. 1. Multiplying a by m, the Product is am, to which prefixing the Sign $\sqrt{\ }$, we have $\sqrt{am}$ the Product required.

Exam. 2. Multiplying mn by d, the Product is mnd, to which prefixing the Sign $\sqrt{\ }$, we have $\sqrt{mnd}$, the Product required.

Exam. 3. Multiplying py by z, the Product is pyz, to which prefixing the Sign $\sqrt{\ }$, we have $\sqrt{pyz}$, the Product required.

Exam. 4. Multiplying zx by a, the Product is zxa, to which prefixing the Sign $\sqrt{\ }$, we have $\sqrt{zxa}$, the Product required.

	Exam. 5.	*Exam.* 6.
Multiply	$\sqrt{pa}$	$\sqrt{zy}$
By	$\sqrt{z}$	$\sqrt{x}$
Product	$\sqrt{paz}$	$\sqrt{zyx}$

	Exam. 7.	*Exam.* 8.
Multiply	$\sqrt{a+b}$	$\sqrt{mn-z}$
By	$\sqrt{y}$	$\sqrt{a}$
Product	$\sqrt{ay+yb}$	$\sqrt{mna-az}$

Exam. 7. Multiplying $a+b$ by y, the Product is $ay+yb$, by Art. 10. to which prefixing the Sign $\sqrt{\ }$, and drawing it over all the Quantities, we have $\sqrt{ay+yb}$, the Product required.

Exam.

Exam. 8. Multiplying $mn - z$ by a the Product is $mna - az$, by Art. 10 and 16. to which prefixing the radical Sign as in the last Example, we have $\sqrt{mna - az}$ the Product required.

	Exam. 9.	*Exam.* 10.	*Exam.* 11.
Multiply	$\sqrt{ap + z}$	$\sqrt{az - ap}$	$\sqrt{d - y}$
By	$\sqrt{y}$	$\sqrt{m}$	$\sqrt{p}$
Product	$\sqrt{apy + yz}$	$\sqrt{azm - apm}$	$\sqrt{dp - py}$

Exam. 9. Multiplying $ap + z$ by y, the Product is $apy + yz$, to which prefixing the radical Sign, we have $\sqrt{apy + yz}$ the Product required.

Exam. 10. Multiplying $az - ap$ by m, the Product is $azm - apm$, to which prefixing the radical Sign, we have $\sqrt{azm - apm}$ the Product required.

Exam. 11. Multiplying $d - y$ by p, the Product is $dp - py$, to which prefixing the radical Sign, we have $\sqrt{dp - py}$ the Product required.

Multiply	$\sqrt{ab}$	$\sqrt{zy}$	$\sqrt{ap - z}$
By	$\sqrt{a - p}$	$\sqrt{d + y}$	$\sqrt{d}$
Product	$\sqrt{aab - abp}$	$\sqrt{zyd + zyy}$	$\sqrt{apd - dz}$
Multiply	$\sqrt{ap + z}$	$\sqrt{ay}$	$\sqrt{m}$
By	$\sqrt{m}$	$\sqrt{d - z}$	$\sqrt{a - py}$
Product	$\sqrt{apm + zm}$	$\sqrt{ayd - ayz}$	$\sqrt{ma - mpy}$

40. *Case* 2. When there are rational Quantities joined to the Surds, then multiply the rational Quantities together as in Multiplication of rational Quantities, after which multiply the Surd Quantities together by the last *Article*, and joining these two Products, this will be the Product required.

If there are no rational Quantities prefixt, then *Unity*, or 1, is always supposed to be the rational Quantity.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Multiply	$a\sqrt{m}$	$ap\sqrt{z}$	$3\sqrt{mn}$	$\sqrt{mp}$
By	$d\sqrt{y}$	$2\sqrt{a}$	$a\sqrt{p}$	$y\sqrt{d}$
Product	$ad\sqrt{my}$	$2ap\sqrt{za}$	$3a\sqrt{mnp}$	$y\sqrt{mpd}$

Exam.

Exam. 1. Multiplying the rational Quantities a and d, the Product is ad, and multiplying the Surds $\sqrt{m}$ by $\sqrt{y}$, the Product is $\sqrt{my}$ by Art. 39. joining this to the rational Quantity ad, we have $ad\sqrt{my}$, the Product required.

Exam. 2. Multiplying the rational Quantities ap by 2, the Product is $2ap$, and multiplying the Surds $\sqrt{z}$ by $\sqrt{a}$, the Product is $\sqrt{za}$ by Art. 39. joining these, and $2ap\sqrt{za}$ is the Product required.

Exam. 3. Multiplying the rational Quantities 3 and a, the Product is $3a$, and multiplying the Surds $\sqrt{mn}$ by $\sqrt{p}$, the Product is $\sqrt{mnp}$, by Art. 39. joining these we have $3a\sqrt{mnp}$, the Product required.

Exam. 4. Multiplying the rational Quantities y and 1, (for 1 is the rational Quantity of $\sqrt{mp}$, there being no rational Quantity prefixt) the Product is y, and multiplying the Surds $\sqrt{mp}$ by $\sqrt{d}$, the Product is $\sqrt{mpd}$ by Art. 39. and joining these we have $y\sqrt{mpd}$, the Product required.

	Exam. 5.	*Exam.* 6.	*Exam.* 7.	*Exam.* 8.
Multiply	$am\sqrt{p}$	$y\sqrt{pq}$	$m\sqrt{2p}$	$2a\sqrt{3z}$
By	$z\sqrt{d}$	$d\sqrt{z}$	$4y\sqrt{y}$	$3d\sqrt{4y}$
Product	$amz\sqrt{pd}$	$yd\sqrt{pqz}$	$4my\sqrt{2py}$	$6ad\sqrt{12zy}$

Exam. 5. The Product of the rational Quantities is amz, and the Product of the Surds is $\sqrt{pd}$, these being joined we have $amz\sqrt{pd}$, the Product required.

Exam. 6. The Product of the rational Quantities is yd, and the Product of the Surds is $\sqrt{pqz}$, these being joined we have $yd\sqrt{pqz}$, the Product required.

Exam. 7. The Product of the rational Quantities is $4my$, and the Product of the Surds is $\sqrt{2py}$, these being joined we have $4my\sqrt{2py}$, the Product required.

Exam. 8. The Product of the rational Quantities is $6ad$, and the Product of the Surds is $\sqrt{12zy}$, these being joined we have $6ad\sqrt{12zy}$, the Product required.

	Exam. 9.	*Exam.* 10.	*Exam.* 11.	*Exam.* 12.
Multiply	$y\sqrt{p}$	$\sqrt{mn}$	$2\sqrt{dx}$	$3\sqrt{2z}$
By	$a\sqrt{z}$	$a\sqrt{y}$	$\sqrt{z}$	$5\sqrt{7y}$
Product	$ya\sqrt{pz}$	$a\sqrt{mny}$	$2\sqrt{dxz}$	$15\sqrt{14yz}$

	Exam. 13.	*Exam.* 14.	*Exam.* 15.
Multiply	$m\sqrt{a+y}$	$d\sqrt{m-pz}$	$a\sqrt{ap+z}$
By	$a\sqrt{p}$	$y\sqrt{d}$	$x\sqrt{y}$
Product	$ma\sqrt{ap+py}$	$dy\sqrt{md-pzd}$	$ax\sqrt{apy+yz}$

Exam. 13. Multiplying the rational Quantities m and a, the Product is ma, and multiplying $a+y$ by p, the Product is $ap+py$, but prefixing to this the Sign $\sqrt{}$, becaufe they are Surds, we have $\sqrt{ap+py}$ for the Product of the Surds, which joining to ma the Product of the rational Quantities, we have $ma\sqrt{ap+py}$, the Product required.

Exam. 14. Multiplying the rational Quantities d and y, the Product is dy, and multiplying the Surds $\sqrt{m-pz}$ by $\sqrt{d}$, the Product is $\sqrt{md-pzd}$, which being joined to dy, the Product of the rational Quantities, we have $dy\sqrt{md-pzd}$, the Product required.

Exam. 15. The Product of the rational Quantities is ax, and the Product of the Surds is $\sqrt{apy+yz}$, thefe being joined we have $ax\sqrt{apy+yz}$, the Product required.

	Exam. 16.	*Exam.* 17.
Multiply	$am\sqrt{py+d}$	$2\sqrt{am-y}$
By	$y\sqrt{z}$	$a\sqrt{p}$
Product	$amy\sqrt{pyz+zd}$	$2a\sqrt{amp-py}$

	Exam. 18.
Multiply	$m\sqrt{pd}$
By	$a\sqrt{d-a}$
Product	$ma\sqrt{pdd-pda}$

Exam. 16. The rational Quantities am and y being multiplied, the Product is amy, and $py+d$ being multiplied by z, the Product is $pyz+zd$; but before it prefix the radical Sign, becaufe thefe Quantities are Surds, then it is $\sqrt{pyz+zd}$, joining this to the Product amy, we have $amy\sqrt{pyz+zd}$, the Product required.

Exam. 17. The Product of the rational Quantities 2 and a, is $2a$, and the Product of the Surds is $\sqrt{amp - py}$: joining these we have $2a\sqrt{amp - py}$, the Product required.

Exam. 18. The Product of the rational Quantities m and a, is ma, and pd multiplied into $d - a$, is $pdd - pda$, to which prefix the radical Sign, because these are Surds, and this becomes $\sqrt{pdd - pda}$, now joining it to ma, we have $ma\sqrt{pdd - pda}$, the Product required.

Multiply	$a\sqrt{p - y}$	$3\sqrt{m - n}$	$\sqrt{n + b}$
By	$2b\sqrt{m}$	$2\sqrt{5a}$	$a\sqrt{z}$
Product	$2a\sqrt{bpm - my}$	$6\sqrt{5ma - 5na}$	$a\sqrt{zn + zb}$

Multiply	$2a\sqrt{3y + z}$	$y\sqrt{p - z}$
By	$b\sqrt{d}$	$3m\sqrt{a}$
Product	$2ab\sqrt{3yd + dz}$	$3ym\sqrt{pa - az}$

Multiply	$5\sqrt{y - x}$
By	$3a\sqrt{2b}$
Product	$15a\sqrt{2by - 2bx}$

Division of Surd Quantities, in which there are two Cases.

41. *Case* 1. When there are no rational Quantities joined with the Surd Quantities, reject all those Quantities in the Dividend and Divisor that are alike, as at Art. 20. and set down the Remainder, to which prefix the radical Sign, and this will be the Quotient sought.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Divide	$\sqrt{mn}$	$\sqrt{ma}$	$\sqrt{abd}$	$\sqrt{abd}$
By	$\sqrt{m}$	$\sqrt{a}$	$\sqrt{ab}$	$\sqrt{a}$
Quotient	$\sqrt{n}$	$\sqrt{m}$	$\sqrt{d}$	$\sqrt{bd}$

Exam. 1. Because m is in both Dividend and Divisor, reject it, and place down n the remaining Part of the Dividend, to which prefixing the radical Sign, and $\sqrt{n}$ is the Quotient required.

Exam.

Exam. 2. Becauſe a is in both Dividend and Diviſor, reject it, and place down m the remaining Part of the Dividend, to which prefixing the radical Sign, we have $\sqrt{m}$, the Quotient required.

Exam. 3. Becauſe ab is in both Dividend and Diviſor, reject it, and place down d the remaining Part of the Dividend, to which prefixing the radical Sign, we have $\sqrt{d}$, the Quotient required.

Exam. 4. Becauſe a is in both Dividend and Diviſor, reject it, and place down bd the remaining Part of the Dividend, to which prefixing the radical Sign, we have $\sqrt{bd}$, the Quotient required.

	Exam. 5.	*Exam.* 6.	*Exam.* 7.	*Exam.* 8.
Divide	$\sqrt{mdy}$	$\sqrt{bzd}$	$\sqrt{bzd}$	$\sqrt{ypa}$
By	$\sqrt{y}$	$\sqrt{bd}$	$\sqrt{zd}$	$\sqrt{yp}$
Quotient	$\sqrt{md}$	$\sqrt{z}$	$\sqrt{b}$	$\sqrt{a}$

Exam. 5. Becauſe y is in both Dividend and Diviſor, reject it, and place down md with the Sign $\sqrt{}$ before it, and $\sqrt{md}$ is the Quotient required.

Exam. 6. Becauſe bd is in both Dividend and Diviſor, reject it, and place down z with the Sign $\sqrt{}$ before it, and $\sqrt{z}$ is the Quotient required.

Exam. 7. Becauſe zd is in both Dividend and Diviſor, reject it, and place down b with the Sign $\sqrt{}$ before it, and we have $\sqrt{b}$, the Quotient required.

Exam. 8. Becauſe yp is in both Dividend and Diviſor, reject it, and place down a with the Sign $\sqrt{}$ before it, and $\sqrt{a}$ is the Quotient required.

	Exam. 9.	*Exam.* 10.	*Exam.* 11.	*Exam.* 12.
Divide	$\sqrt{max}$	$\sqrt{ndy}$	$\sqrt{ayz}$	$\sqrt{abd}$
By	$\sqrt{x}$	$\sqrt{ny}$	$\sqrt{a}$	$\sqrt{ad}$
Quotient	$\sqrt{ma}$	$\sqrt{d}$	$\sqrt{yz}$	$\sqrt{b}$

	Exam. 13.	*Exam.* 14.	*Exam.* 15.
Divide	$\sqrt{am+ap}$	$\sqrt{py-pn}$	$\sqrt{bd-bm}$
By	$\sqrt{a}$	$\sqrt{p}$	$\sqrt{b}$
Quotient	$\sqrt{m+p}$	$\sqrt{y-n}$	$\sqrt{d-m}$

Exam.

Exam. 13. If we divide $am+ap$ by a, the Quotient is $m+p$ by Art. 22. but becauſe they are Surds, prefix the Sign $\surd$ to $m+p$, and $\sqrt{m+p}$ is the Quotient required.

Exam. 14. Dividing $py-pn$ by p, the Quotient is $y-n$, by Art. 22 and 24. to which prefixing the Sign $\surd$, we have $\sqrt{y-n}$, the Quotient required.

Exam. 15. Dividing $bd-bm$ by b, the Quotient is $d-m$, by Art. 22 and 24. to which prefixing the Sign $\surd$, we have $\sqrt{d-m}$, the Quotient required.

Divide	$\sqrt{bn+ba}$	$\sqrt{mx-md}$	$\sqrt{nz-np}$
By	$\sqrt{b}$	$\sqrt{m}$	$\sqrt{n}$
Quotient	$\sqrt{n+a}$	$\sqrt{x-d}$	$\sqrt{z-p}$

Divide	$\sqrt{zx-zy}$	$\sqrt{ad+ay}$	$\sqrt{bd-bm}$
By	$\sqrt{z}$	$\sqrt{a}$	$\sqrt{b}$
Quotient	$\sqrt{x-y}$	$\sqrt{d+y}$	$\sqrt{d-m}$

The Truth of theſe Operations are proved by multiplying the Quotient by the Diviſor, for if that produces the Dividend, the Work is true, otherwiſe it is erroneous. Thus in Example 2, Page 68. the Diviſor is $\sqrt{a}$, and the Quotient is $\sqrt{m}$, which being multiplied by Art. 39. the Product is $\sqrt{ma}$, the given Dividend.

And at Example 6, Page 69. the Diviſor is $\sqrt{bd}$, and the Quotient is $\sqrt{z}$, which being multiplied by Art. 39. the Product is $\sqrt{bzd}$, the given Dividend.

And at Example 13, the Diviſor is $\sqrt{a}$, and the Quotient is $\sqrt{m+p}$, which being multiplied by Art. 39. the Product is $\sqrt{am+ap}$, the given Dividend; in the ſame Manner may any of the other Examples be proved.

42. *Caſe* 2. When there are rational Quantities joined with the Surds, divide the rational Quantities by the rational Quantities, by the Rules in Diviſion of rational Quantities; and to their Quotient, join the Quotient of the Surds found by the laſt Article, which will be the Quotient required.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Divide	$ay\sqrt{mn}$	$bm+yz$	$yd\sqrt{az}$	$ma\sqrt{ayn}$
By	$a\sqrt{m}$	$m\sqrt{z}$	$y\sqrt{a}$	$a\sqrt{ay}$
Quotient	$y\sqrt{n}$	$b\sqrt{y}$	$d\sqrt{z}$	$m\sqrt{n}$

Exam.

Exam. 1. Dividing the rational Quantities ay by a, the Quotient is y by Art. 20. and dividing $\sqrt{mn}$ by $\sqrt{m}$, the Quotient is $\sqrt{n}$ by Art. 41. now joining y to $\sqrt{n}$, we have $y\sqrt{n}$, the Quotient required.

Exam. 2. Dividing the rational Quantities bm by m, the Quotient is b by Art. 20. and dividing $\sqrt{yz}$ by $\sqrt{z}$, the Quotient is $\sqrt{y}$ by Art 41. now joining b and $\sqrt{y}$, we have $b\sqrt{y}$, the Quotient required.

Exam. 3. Dividing the rational Quantities yd by y, the Quotient is d by Art. 20. and dividing $\sqrt{az}$ by $\sqrt{a}$, the Quotient is $\sqrt{z}$ by Art. 41. now joining d and $\sqrt{z}$, we have $d\sqrt{z}$, the Quotient required.

Exam. 4. Dividing the rational Quantities ma by a, the Quotient is m by Art. 20. and dividing $\sqrt{ayn}$ by $\sqrt{ay}$, the Quotient is $\sqrt{n}$ by Art. 41. now joining m and $\sqrt{n}$, we have $m\sqrt{n}$, the Quotient required.

	Exam. 5.	*Exam.* 6.	*Exam.* 7.	*Exam.* 8.
Divide	$ayn\sqrt{mn}$	$mn\sqrt{xay}$	$xa\sqrt{nd}$	$dz\sqrt{anp}$
By	$ay\sqrt{m}$	$n\sqrt{xy}$	$a\sqrt{n}$	$z\sqrt{an}$
Quotient	$n\sqrt{n}$	$m\sqrt{a}$	$x\sqrt{d}$	$d\sqrt{p}$

Exam. 5. Dividing the rational Quantities ayn by ay, the Quotient is n by Art. 20. and dividing $\sqrt{mn}$ by $\sqrt{m}$, the Quotient is n by Art. 41. now joining n and $\sqrt{n}$, we have $n\sqrt{n}$, the Quotient required.

Exam. 6. Dividing the rational Quantities mn by n, the Quotient is m by Art. 20. and dividing $\sqrt{xay}$ by $\sqrt{xy}$, the Quotient is $\sqrt{a}$ by Art. 41. now joining m and $\sqrt{a}$, we have $m\sqrt{a}$, the Quotient required.

Exam. 7. Dividing the rational Quantities xa by a, the Quotient is x, and dividing $\sqrt{nd}$ by $\sqrt{n}$, the Quotient is $\sqrt{d}$ by Art. 41. now joining x and $\sqrt{d}$, we have $x\sqrt{d}$, the Quotient required.

Exam. 8. Dividing the rational Quantities dz by z, the Quotient is d, and dividing $\sqrt{anp}$ by $\sqrt{an}$, the Quotient is $\sqrt{p}$ by Art. 41. now joining d and $\sqrt{p}$, we have $d\sqrt{p}$, the Quotient required.

	Exam. 9.	*Exam.* 10.	*Exam.* 11.	*Exam.* 12.
Divide	$4mn\sqrt{ab}$	$my\sqrt{az}$	$dn\sqrt{xy}$	$8an\sqrt{rd}$
By	$2m\sqrt{a}$	$y\sqrt{z}$	$n\sqrt{x}$	$4a\sqrt{r}$
Quotient	$2n\sqrt{b}$	$m\sqrt{a}$	$d\sqrt{y}$	$2n\sqrt{d}$

Exam.

	Exam. 13.	*Exam.* 14.	*Exam.* 15.	*Exam.* 16.
Divide	$mx\sqrt{pq}$	$4an\sqrt{rd}$	$xz\sqrt{myp}$	$rm\sqrt{dz}$
By	$x\sqrt{p}$	$a\sqrt{d}$	$z\sqrt{my}$	$r\sqrt{d}$
Quotient	$m\sqrt{q}$	$4n\sqrt{r}$	$x\sqrt{p}$	$m\sqrt{z}$

	Exam. 17.	*Exam.* 18.	*Exam.* 19.
Divide	$mn\sqrt{ap+ax}$	$yp\sqrt{zd+zm}$	$day\sqrt{ym+yr}$
By	$m\sqrt{a}$	$p\sqrt{z}$	$da\sqrt{y}$
Quotient	$n\sqrt{p+x}$	$y\sqrt{d+m}$	$y\sqrt{m+r}$

Exam. 17. Dividing the rational Quantities mn by m, the Quotient is n by Art. 20. and dividing $\sqrt{ap+ax}$ by $\sqrt{a}$, we have $\sqrt{p+x}$ by Art. 41. and joining n and $\sqrt{p+x}$, we have $n\sqrt{p+x}$, the Quotient required.

Exam. 18. Dividing the rational Quantities yp by p, the Quotient is y by Art. 20. and dividing $\sqrt{zd+zm}$ by $\sqrt{z}$, the Quotient is $\sqrt{d+m}$ by Art. 41. joining y and $\sqrt{d+m}$, we have $y\sqrt{d+m}$, the Quotient required.

Exam. 19. Dividing the rational Quantities day by da, the Quotient is y by Art. 20. and dividing $\sqrt{ym+yr}$ by $\sqrt{y}$, the Quotient is $\sqrt{m+r}$ by Art. 41. joining y and $\sqrt{m+r}$, we have $y\sqrt{m+r}$, the Quotient required. The following Examples are done in the fame Manner.

Divide	$4an\sqrt{dy+dn}$	$an\sqrt{pz-pb}$	$6bdy\sqrt{pm+pd}$
By	$2a\sqrt{d}$	$a\sqrt{p}$	$3bd\sqrt{p}$
Quotient	$2n\sqrt{y+n}$	$n\sqrt{z-b}$	$2y\sqrt{m+d}$

Divide	$pn\sqrt{dx-db}$	$12ba\sqrt{py-px}$	$anx\sqrt{pd-pm}$
By	$n\sqrt{d}$	$3a\sqrt{p}$	$ax\sqrt{p}$
Quotient	$p\sqrt{x-b}$	$4b\sqrt{y-x}$	$n\sqrt{d-m}$

The Truth of thefe Operations are proved likewife from multiplying the Quotient by the Divifor, and if that Product makes the Dividend, the Work is true, if not, there is a Miftake. Thus in

Exam.

Exam. 1. The Quotient is $y\sqrt{n}$, and the Divisor is $a\sqrt{m}$; now multiplying $y\sqrt{n}$ by $a\sqrt{m}$, by Art. 40. first multiply the rational Quantities y and a, this Product is ay, and multiplying $\sqrt{n}$ by $\sqrt{m}$, this Product is $\sqrt{mn}$, and joining this to ay we have $ay\sqrt{mn}$ the Product, which being the same as the given Dividend, proves the Work to be true.

And at *Exam.* 5. the Divisor is $ay\sqrt{m}$, the Quotient is $n\sqrt{n}$, now multiplying $ay\sqrt{m}$ by $n\sqrt{n}$, according to Art. 40. we first multiply the rational Quantities ay by n, and this Product is ayn; then multiplying $\sqrt{m}$ by $\sqrt{n}$, this Product is $\sqrt{mn}$, and joining this to ayn, the Product is $ayn\sqrt{mn}$, which being the same with the given Dividend, the Work is true.

And at *Exam.* 17. the Divisor is $m\sqrt{a}$, and the Quotient is $n\sqrt{p+x}$, and multiplying these by Art. 40. we first multiply the rational Quantities m and n together, and this Product is mn, then multiplying $\sqrt{a}$ by $\sqrt{p+x}$, this Product is $\sqrt{ap+ax}$, which being joined to mn, the Product is $mn\sqrt{ap+ax}$, the same as the given Dividend; and so may any of the other Examples be proved.

Involution of Surd Quantities.

43. *Case* 1. When there are no rational Quantities joined with the Surds, it is only setting the Quantities down without their radical Sign, which raises the given Root as high as is the Index of the radical Sign.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Raise to the Square or second Power	$\sqrt{a}$	$\sqrt{mn}$	$\sqrt{na}$	$\sqrt{b}$
The Square	a	mn	na	b

This being no more, according to the Rule, but to set down the Quantities without their radical Sign, it is so easy as not to want any farther Explanation.

The Reason on which the Operation is founded, is, that any Quantity or Number being multiplied by itself, will produce the Square of that Quantity or Number, thus $2 \times 2 = 4$, whence 4 is the Square of 2, and $a \times a = aa$, which is the Square of a, and so of any other Quantity. Now supposing the Square Root of a was to be extracted, which by Art. 33. is $\sqrt{a}$. But

as $\sqrt{a}$ is the Root, and a was the Square from which that Root was extracted, hence $\sqrt{a}$ multiplied into $\sqrt{a}$, muſt produce a, by what has been juſt ſaid: Now $\sqrt{a}$ multiplied by $\sqrt{a}$, is $\sqrt{aa}$ by Art. 39. and as $\sqrt{aa}$ ſignifies the Square Root of aa, which is a by Art. 33. it follows, that to involve any Surd that has no rational Quantities joined with it, is only to ſet down the Quantities without their radical Sign.

To find the Square or ſecond Power of	$\sqrt{ax}$	$\sqrt{nd}$	$\sqrt{pr}$	$\sqrt{z}$
The Square	ax	nd	pr	z

And if there are ſeveral Quantities connected by the Signs + or —, and are all under the radical Sign, they are involved in the ſame Manner.

Raiſe to the ſecond Power or Square	$\sqrt{a+b}$	$\sqrt{an-d}$	$\sqrt{p+ny}$
The Square	$a+b$	$an-d$	$p+ny$
Raiſe to the ſecond Power or Square	$\sqrt{pd-n}$	$\sqrt{dz+zy}$	$\sqrt{pm-nd}$
The Square	$pd-n$	$dz+zy$	$pm-nd$
Raiſe to the 2d Power or Square	$\sqrt{a+y-d}$	$\sqrt{am-n+db}$	$\sqrt{pz+zx-xd}$
The Square	$a+y-d$	$am-n+db$	$pz+zx-xd$

44. *Caſe* 2. When there are rational Quantities joined with the Surds, then involve the rational Quantities as high as is the Index of the Surd, and multiply theſe involved Quantities into the Surd Quantities, after the radical Sign is taken away.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
Raiſe to the Square	$a\sqrt{m}$	$b\sqrt{nz}$	$d\sqrt{y}$	$zz\sqrt{b}$
The Square	aam	$bbnz$	ddy	$zzzzb$

Ex. 1. The rational Quantity a ſquared is by Art. 31.	aa
The Surd Quantity $\sqrt{m}$ being put down without the radical Sign is — —	m
Theſe being multiplied, the Product is the Square required	aam

Exam.

Ex. 2. The rational Quantity b squared is by Art. 31. bb
The Surd Quantity $\sqrt{nz}$ without the radical Sign is nz
These multiplied, the Product is the Square required $bbnz$

Exam. 3. The rational Quantity d squared is - dd
The Surd Quantity $\sqrt{y}$ without the radical Sign is y
These multiplied, the Product is the Square required ddy

Exam. 4. The rational Quantity zz squared is $zzzz$
The Surd Quantity $\sqrt{b}$ without the radical Sign is b
These multiplied, the Product is the Square required $zzzzb$

	Exam. 5.	*Exam.* 6.	*Exam.* 7.	*Exam.* 8.
Raise to the Square	$an\sqrt{p}$	$dz\sqrt{yx}$	$p\sqrt{xy}$	$da\sqrt{z}$
The Square	$aannp$	$ddzzyx$	$ppxy$	$ddaaz$

Exam. 5. The rational Quantity an squared is $aann$
The Surd Quantity $\sqrt{p}$ without the radical Sign is p
These multiplied, the Product is the Square required $aannp$

Exam. 6. The rational Quantity dz squared is $ddzz$
The Surd Quantity $\sqrt{yx}$ without the radical Sign is yx
These multiplied, the Product is the Square required $ddzzyx$

Exam. 7. The rational Quantity p squared is - pp
The Surd Quantity $\sqrt{xy}$ without the radical Sign is xy
These multiplied, the Product is the Square required $ppxy$

Exam. 8. The rational Quantity da squared is $ddaa$
The Surd Quantity $\sqrt{x}$ without the radical Sign is z
These multiplied, the Product is the Square required $ddaaz$

Raise to the Square	$m\sqrt{pz}$	$mn\sqrt{d}$	$a\sqrt{rd}$	$py\sqrt{m}$
The Square	$mmpz$	$mmnnd$	$aard$	$ppyym$
Raise to the Square	$x\sqrt{pd}$	$xn\sqrt{a}$	$z\sqrt{px}$	$az\sqrt{d}$
The Square	$xxpd$	$xxnna$	$zzpx$	$aazzd$

And if there are more Quantities than one under the radical Sign, connected with the Signs + or —, then after the rational Quantities are involved, or raised as high as is the Index of the

 Surd;

Surd; place these under the radical Quantities, without their Sign, then multiply them by the Rule of Multiplication at Art. 10. &c. and this will be the Square required.

	Exam. 1.	*Exam.* 2.	*Exam.* 3.
Raise to the Square	$a\sqrt{m+y}$	$b\sqrt{d+z}$	$m\sqrt{z-x}$
The Square is	$aam+aay$	$bbd+bbz$	$mmz-mmx$

Exam. 1. The Surd Quantity $\sqrt{m+y}$ without the radical Sign is - - $m+y$
The rational Quantity a squared is - - aa
These being multiplied according to Art. 10. the Product is the Square required $aam+aay$

Exam. 2. The Surd Quantity $\sqrt{d+z}$ without the radical Sign is - - $d+z$
The rational Quantity b squared is - - bb
These multiplied, the Prod. is the Square required $bbd+bbz$

Exam. 3. The Surd Quantity $\sqrt{z-x}$ without the radical Sign is - - $z-x$
The rational Quantity m squared is - - mm
These multiplied, the Product is the Square required - $mmz-mmx$

	Exam. 4.	*Exam.* 5.	*Exam.* 6.
Raise to the Square	$z\sqrt{a+n}$	$x\sqrt{b-d}$	$d\sqrt{z+y}$
The Square	$zza+zzn$	$xxb-xxd$	$ddz+ddy$

Exam. 4. The Surd Quantity $\sqrt{a+n}$ without the radical Sign is - - $a+n$
The rational Quantity z squared is - - zz
These multiplied, the Product is the Square required - $zza+zzn$

Exam. 5. The Surd Quantity $\sqrt{b-d}$ without the radical Sign is - - $b-d$
The rational Quantity x squared is - - xx
These multiplied, the Product is the Square required - $xxb-xxd$

Exam.

Exam. 6. The Surd Quantity $\sqrt{z+y}$ without the radical Sign is	$z+y$
The rational Quantity d squared is	dd
These multiplied, the Product is the Square required	$ddz+ddy$

Raise to the Square	$y\sqrt{a-n}$	$n\sqrt{a+d}$	$d\sqrt{p-z}$
The Square	$yya-yyn$	$nna+nnd$	$ddp-ddz$

Raise to the Square	$e\sqrt{p-r}$	$d\sqrt{e+y}$	$z\sqrt{n-y}$
The Square	$eep-eer$	$dde+ddy$	$zzn-zzy$

45. *Case* 3. But if there are rational Quantities connected with the Surd Quantities by the Signs $+$ or $-$, they are involved in the same Manner as compound Quantities, at Art. 32. carefully observing the Directions concerning the Multiplication of Surd Quantities, at Art. 40. and their Involution at Art. 43.

To raise to the Square or second Power	$a+\sqrt{b}$
Putting down again the same Quantity	$a+\sqrt{b}$
Now multiply $a+\sqrt{b}$ by a, and a multiplied by a, the Product is aa, and $\sqrt{b}$ multiplied by a, the Product is $a\sqrt{b}$, by Art. 40. therefore $a+\sqrt{b}$ multiplied by a is	$aa+a\sqrt{b}$
Then multiply $a+\sqrt{b}$, by $\sqrt{b}$, and a multiplied by $\sqrt{b}$, the Prod. is $a\sqrt{b}$ by Art. 40. and $\sqrt{b}$ multip. by $\sqrt{b}$, the Prod. is b, by Art. 43. whence $a+\sqrt{b}$ multiplied by $\sqrt{b}$ is	$a\sqrt{b}+b$
The Sum is $aa+2a\sqrt{b}+b$: for $a\sqrt{b}$ added to $a\sqrt{b}$ is $2a\sqrt{b}$, by Art. 35. whence the Square of $a+\sqrt{b}$ is	$aa+2a\sqrt{b}+b$

To raise to the Square or second Power	$d+\sqrt{z}$
Putting down again the same Quantity	$d+\sqrt{z}$
The Product from multiplying $d+\sqrt{z}$ by d, by what is mentioned in the last Example is	$dd+d\sqrt{z}$
The Product from multiplying $d+\sqrt{z}$ by $\sqrt{z}$, by what is mentioned in the last Example is	$d\sqrt{z}+z$
The Sum added as in the last Example is the Square of $d+\sqrt{z}$	$dd+2d\sqrt{z}+z$

To

To raise to the Square or second Power	$x - \sqrt{a}$
Putting down again the same Quantity	$x - \sqrt{a}$
The Prod. from multiplying $x - \sqrt{a}$ by x, for x multiplied by x, the Prod. is xx, and $-\sqrt{a}$ multiplied by x, the Prod. is $-x\sqrt{a}$, by Art. 40. the Signs of $-\sqrt{a}$ and x being different	$xx - x\sqrt{a}$
The Product from multiplying $x - \sqrt{a}$ by $-\sqrt{a}$, for x multiplied by $-\sqrt{a}$, the Product is $-x\sqrt{a}$, the Signs being unlike, but $-\sqrt{a}$ multiplied by $-\sqrt{a}$, the Signs being alike, the Product is $\sqrt{aa}$ or a by Art. 39 and 43.	$-x\sqrt{a} + a$
Their Sum is the Square of $x - \sqrt{a}$	$xx - 2x\sqrt{a} + a$

To raise to the Square or second Power	$y - \sqrt{x}$
Putting down again the same Quantity	$y - \sqrt{x}$
The Product from multiplying $y - \sqrt{x}$ by y, from what is mentioned in the last Example is	$yy - y\sqrt{x}$
The Product from multiplying $y - \sqrt{x}$ by $-\sqrt{x}$, from what is said in the last Example is	$-y\sqrt{x} + x$
Their Sum is the Square of $y - \sqrt{x}$	$yy - 2y\sqrt{x} + x$

To raise to the Square or second Power	$b + \sqrt{xa}$
Putting down the same Quantity	$b + \sqrt{xa}$
Multiplying $b + \sqrt{xa}$ by b we have	$bb + b\sqrt{xa}$
Multiplying $b + \sqrt{xa}$ by $\sqrt{xa}$ we have	$b\sqrt{xa} + xa$
The Sum being the Square of $b + \sqrt{xa}$,	$bb + 2b\sqrt{xa} + xa$

To raise to the Square or second Power	$m + \sqrt{dz}$
Putting down the same Quantity	$m + \sqrt{dz}$
Multiplying $m + \sqrt{dz}$ by m we have	$mm + m\sqrt{dz}$
Multiplying $m + \sqrt{dz}$ by $\sqrt{dz}$ we have	$m\sqrt{dz} + dz$
The Sum being the Square of $m + \sqrt{dz}$,	$mm + 2m\sqrt{dz} + dz$

To raise to the Square or second Power	$z - \sqrt{dn}$
Putting down the same Quantity	$z - \sqrt{dn}$
	$zz - z\sqrt{dn}$
	$-z\sqrt{dn} + dn$
The Square of $z - \sqrt{dn}$	$zz - 2z\sqrt{dn} + dn$

To raise to the Square or second Power -	$p - \sqrt{yz}$
Putting down again the same Quantity -	$p - \sqrt{yz}$
	$pp - p\sqrt{yz}$
	$-p\sqrt{yz} + yz$
The Square of $p - \sqrt{yz}$ -	$pp - 2p\sqrt{yz} + yz$

Of EQUATIONS.

HAVING thus copiously explained all the Rules necessary to be known, in order to the Solution of Questions, we come now to their Use and Application in the Reduction of Equations, or the Method by which Problems are solved, and Questions answered.

When any Problem or Question is proposed to be answered *Algebraically*, for the several Numbers that are in the Question we generally put Letters, representing likewise the Numbers which are to be found by Letters, and for Distinction Sake use the *Vowels* for the unknown Numbers, or those that are to be found, and *Consonants* for those that are known, or given.

Then we begin to express all the Conditions of the Question, by ranging and connecting the Letters, by Help of the foregoing Signs, in such a Manner that they shall represent all the Circumstances of the Question, this being only to translate the Question from *English* into *Algebra*.

Thus if the Proposition, that 6 being added to 5, the Sum is equal to 11, was to be expressed in *Algebra*.

Now suppose $b = 6$, $d = 5$, $m = 11$.

Then the above Proposition will be expressed thus, $b + d = m$.

And when any Letters or Numbers are so connected, that between any of them there appears this Sign $=$, it is called an Equation, for the Sign $=$ signifies *Equality* or *Equation*, and in the due ordering and managing these Equations consists the whole of the *Analytic* Science, or *Algebra*.

Equations consist of Quantities or Letters, some known, and others unknown, and the grand Work is so to manage the Equations, that express what is given in the Question, by the Rules of Certainty and Science, that all the known Quantities may at last be found on one Side of the Equation, and the unknown Quantity by itself on the other of the Equation: For when this is done, the Equation is brought to a Solution, and the Question is answered.

And

And that Part of *Algebra* which teaches how to manage theſe Quantities, ſo as to carry all the known Quantities on one Side, leaving the unknown Quantity by itſelf on the other Side of the Equation, is called the *Reduction of Equations*, which is done by *Addition*, *Subſtraction*, *Multiplication*, *Diviſion*, *Involution*, and *Evolution*, according as the Caſe requires.

To reduce an Equation by Addition, or Subſtraction.

46. WHEN any known Quantities are on the ſame Side of the Equation with the unknown Quantity, and connected by the Signs $+$ or $-$, to reduce ſuch an Equation is only to *tranſpoſe or place the known Quantities on the other Side of the Equation, or Sign of Equality, prefixing to them their contrary Sign*, that is, *thoſe Quantities which have the Sign* $+$, *after they are tranſpoſed muſt have the Sign* $-$, *and thoſe which have the Sign* $-$ *muſt have the Sign* $+$.

Queſtion 1. *To find that Number to which* 6 *being added, and ſubſtracting* 15 *from this Sum, the Remainder may be equal to* 11.

Now ſuppoſe $a =$ the Number ſought, $b = 6$, $d = 15$, $m = 11$.

Then I am to find a Number, which call	1	a
To which 6 or b being added, it is by Art. 6.	2	$a + b$
From which Sum 15 or d is to be ſubſtracted, that is, to $a + b$ connect d by the Sign $-$, then it is	3	$a + b - d$
Which $a + b - d$ is to be equal to 11 or m, that is	4	$a + b - d = m$
Now to reduce this Equation, or to anſwer the Queſtion, I obſerve d, a known Quantity, is on the ſame Side of the Equation with the unknown Quantity a, therefore tranſpoſe d, that is, write down the remaining Part of that Side of the Equation without d, and place it on the other Side with the Sign $+$, it having before the Sign $-$, then we have	5	$a + b = m + d$
Again b is a known Quantity on the ſame Side of the Equation with a, then by taking it away from that Side of the Equation, and placing it on the other Side with the contrary Sign, or $-$, we have	6	$a = m + d - b$

Here

Here the Queſtion is ſolved, for the unknown Number or Quantity a, is equal to the Number repreſented by m, added to the Number repreſented by d, from which Sum ſubſtracting the Number repreſented by b.

11	repreſented by m
15	repreſented by d
26	Sum of the Numbers repreſented by m and d
6	repreſented by b, to be ſubſtracted
Remains 20	which is a, or the Number ſought.

And that this is the Number required, is thus proved, from the Conditions of the Queſtion.

I ſay the Number ſought is — —	20
For if to this is added — — —	6
The Sum is — — —	26
From which ſubſtracting — —	15
There remains as the Queſtion required —	11

Queſtion 2. *A Man being aſked how many Shillings he had, ſaid, if you add* 15 *to their Number, and then ſubſtract* 20 *from that Sum, and then add* 19 *to the Remainder, I ſhall have* 64 *Shillings. How many Shillings had he?*

Let $a =$ the Number of Shillings ſought, $b = 15$, $d = 20$, $m = 19$, $n = 64$.

Then, A Man had a certain Number of Shillings, which call	1	a
To which 15 or b being added we have, by Art. 6.	2	$a + b$
From which Sum taking away 20 or d, that is, connect d by the Sign —	3	$a + b - d$
To which adding 19 or m we have, by Art. 6.	4	$a + b - d + m$
Which $a + b - d + m$ is to be equal to 64 or n, hence	5	$a + b - d + m = n$
Now to reduce this Equation, or anſwer the Queſtion: I begin with tranſpoſing m a known Quantity, by putting down the remaining Part of that Side of the Equation, and placing m on the other Side with the contrary Sign, which gives	6	$a + b - d = n - m$

 And

And to tranſpoſe d another known Quantity, put down the remaining Part of that Side of the Equation, and d on the other Side with a contrary Sign, whence we have	7	$a+b=n-m+d$
And laſtly, by tranſpoſing b, that is, placing it on the other Side of the Equation with a contrary Sign, we have	8	$a=n-m+d-b$

That is, if from the Number repreſented by n we ſubſtract that repreſented by m, and to the Remainder add the Number repreſented by d, and from this Sum ſubſtract the Number repreſented by b, the Remainder will be the Number ſought.

64 repreſented by n
19 repreſented by m, to be ſubſtracted

45 or $n-m$
20 repreſented by d, to be added

65 or $n-m+d$
15 repreſented by b, to be ſubſtracted

50 the Number ſought or a; and therefore the Man had 50*s.* at firſt, which is thus proved, from the Conditions of the Queſtion.

I ſay he had at firſt — —	50 *s.*
For if to them you add — —	15
	65
And from that Sum ſubſtract — —	20
	45
And then add to the Remainder — —	19
It makes what the Queſtion requires —	64

Queſtion 3. *A Countryman aſked another how many Eggs he had, Why, ſays he, if you ſubſtract* 15 *from their Number, and then add* 21 *to thoſe that are left, and ſubſtract* 7 *from that Sum, but if you add* 19 *to what is then left I ſhall have* 43 *Eggs. How many Eggs had he?*

Let

Let $a =$ the Number of Eggs, $b = 15$, $d = 21$, $m = 7$, $n = 19$, $p = 43$.

Now the Countryman had a Number of Eggs, which call -	1	a
From which 15 or b being ſubſtracted, or connecting b by the Sign —, we have - -	2	$a - b$
To which $a - b$, if we add 21 or d, we have by Art. 6. -	3	$a - b + d$
From which Sum ſubſtracting 7, or connecting m by the Sign —	4	$a - b + d - m$
To which adding 19 or n, we have by Art. 6.	5	$a - b + d - m + n$
And this $a - b + d - m + n$ is to be equal to 43 or p, hence	6	$a - b + d - m + n = p$
Now to reduce this Equation, or anſwer the Queſtion, I begin with tranſpoſing n, by putting down the remaining Part of that Side of the Equation, and n on the other Side with its contrary Sign, then - - -	7	$a - b + d - m = p - n$
Now tranſpoſe m, by putting down the remaining Part of that Side of the Equation, and m on the other Side with its contrary Sign, and we have - -	8	$a - b + d = p - n + m$
Then tranſpoſe d, by putting down the remaining Part of that Side of the Equation, and d on the other Side with its contrary Sign, then	9	$a - b = p - n + m - d$
Laſtly, tranſpoſe b, by putting down the remaining Part of that Side of the Equation, and b on the other Side with its contrary Sign, and it is - -	10	$a = p - n + m - d + b$

Hence a, the unknown Quantity or Number of Eggs, is equal to the Number repreſented by p, ſubſtracting from it the Number repreſented by n, adding to this Remainder the Number repreſented by m, ſubſtracting from this Sum the Number

reprefented by d, and adding to the Remainder the Number reprefented by b.

43	reprefented by p
19	reprefented by n, to be fubftracted
24	or $p - n$
7	reprefented by m, to be added
31	or $p - n + m$
21	reprefented by d, to be fubftracted
10	or $p - n + m - d$
15	reprefented by b, to be added

25 the Number fought or a; and therefore the Man had 25 Eggs, which is thus proved, from the Conditions of the Queftion.

I fay the Man had - -	25 Eggs
For if from them you fubftract - -	15
	10
And to the Remainder add -	21
	31
And from this Sum fubftract -	7
	24
And to the Remainder add - -	19
It makes what the Queftion requires -	43

Queftion 4. *To find that Number to which* 19 *being added, if from that Sum we fubftract* 50, *and add* 7 *to the Remainder, and fubftract* 60 *from this Sum, and by adding* 6 *to that Remainder, this Sum may be* 22.

Let $a =$ Number fought, $b = 19$, $d = 50$, $m = 7$, $n = 60$, $p = 6$, $g = 22$.

Now I am to find a Number, which call - - -	1	a
To which 19 or b being added, we have by Art. 6. - -	2	$a + b$
From which Sum fubftracting 50 or d, that is, connecting d by the Sign —, and it is	3	$a + b - d$

And

And to this Remainder adding 7 or m, we have by Art. 6.	4	$a+b-d+m$
From this Sum ſubſtracting 60 or n, that is, connecting n by the Sign $-$ - -	5	$a+b-d+m-n$
And to this Remainder adding 6 or p, we have - -	6	$a+b-d+m-n+p$
And this $a+b-d+m-n+p$ is to be equal to 22 or g, hence	7	$a+b-d+m-n+p=g$
Now to anſwer the Queſtion, tranſpoſe p, by putting down the remaining Part of that Side of the Equation, and p on the other Side with its contrary Sign, hence - -	8	$a+b-d+m-n=g-p$
Then tranſpoſe n, by putting down the remaining Part of that Side of the Equation, and n on the other Side with its contrary Sign, then -	9	$a+b-d+m=g-p+n$
Then tranſpoſe m, by putting down the remaining Part of that Side of the Equation, and m on the other Side with its contrary Sign, whence	10	$a+b-d=g-p+n-m$
Then tranſpoſe d, by putting down the remaining Part of that Side of the Equation, and d on the other Side with its contrary Sign, and -	11	$a+b=g-p+n-m+d$
Laſtly, tranſpoſe b, by putting down the remaining Part of that Side of the Equation, and b on the other Side with its contrary Sign, we have	12	$a=g-p+n-m+d-b$

Hence a, the unknown Number, is equal to the Number repreſented by g, ſubſtracting from it the Number repreſented by p, adding to the Remainder the Number repreſented by n, ſubſtracting from this Sum the Number repreſented by m, adding to the Remainder the Number repreſented by d, and ſubſtracting from this Sum the Number repreſented by b.

22 the Number reprefented by g
6 the Number reprefented by p, fubftract

16 or $g - p$
60 the Number reprefented by n, add

76 or $g - p + n$
7 the Number reprefented by m, fubftract

69 or $g - p + n - m$
50 the Number reprefented by d, add

119 or $g - p + n - m + d$
19 the Number reprefented by b, fubftract

100 the Number fought or a, which is thus proved, from the Conditions of the Queftion.

I fay the Number fought was - -	100
For if to that you add - - -	19
	119
And from the Sum fubftract - -	50
	69
And to the Remainder add - - -	7
	76
And from the Sum fubftract - -	60
	16
And add to the Remainder - -	6
It makes what the Queftion requires -	22

The Directions to the two following Queftions are not quite fo copious, that the Judgment of the Learner may be a little more exercifed.

Queftion 5. *A Number of Men were walking on a Bowling-Green, one Man afked another how many there were, the other replied, if you fubftract* 7 *from their Number, and add* 15 *to the Remainder, and fubftract* 9 *from the Sum, and add* 56 *to the Remainder, and fubftract* 2 *from that Sum, this will leave* 100. *To find the Number of Men on the Bowling-Green.*

Let $a =$ Number of Men on the Bowling-Green, $b = 7$, $d = 15$, $g = 9$, $m = 56$, $n = 2$, $p = 100$.

I am

I am to find the Number of Men on the Bowling-Green, which call - -	1	a
From which 7 or b being ſubſtracted, which is only to connect b by the Sign —	2	$a - b$
To which Remainder adding 15 or d, we have by Art. 6.	3	$a - b + d$
From which Sum ſubſtracting 9 or g, or connecting g by the Sign —, and we have	4	$a - b + d - g$
To this Remainder adding m or 56, by Art. 6.	5	$a - b + d - g + m$
From which Sum ſubſtracting 2 or n, that is, connecting n by the Sign —, it is	6	$a - b + d - g + m - n$
Which $a - b + d - g + m - n$ is to be equal to 100 or p, hence - -	7	$a - b + d - g + m - n = p$
By tranſpoſing n we have -	8	$a - b + d - g + m = p + n$
By tranſpoſing m we have -	9	$a - b + d - g = p + n - m$
By tranſpoſing g we have -	10	$a - b + d = p + n - m + g$
By tranſpoſing d we have -	11	$a - b = p + n - m + g - d$
By tranſpoſing b we have -	12	$a = p + n - m + g - d + b$

100 is the Number repreſented by p
2 or n, to be added

102 or $p + n$
56 or m, to be ſubſtracted

46 or $p + n - m$
9 or g, to be added

55 or $p + n - m + g$
15 or d, to be ſubſtracted

40 or $p + n - m + g - d$
7 or b, to be added

47 the Number ſought or a, for $a = p + n - m + g - d + b$.

Now to prove 47 was the Number of Men that were on the Bowling-Green, let us try if it will anſwer the Conditions of the Queſtion.

I ſay

I ſay the Number of Men were - -	47
For if from them you ſubſtract - -	7
	40
And add to the Remainder -	15
	55
And from the Sum ſubſtract - -	9
	46
And add to the Remainder - -	56
	102
And from the Sum ſubſtract - -	2
It makes what the Queſtion requires -	100

Queſtion 6. *A Perſon required another to tell him how many Shillings he had, by ſaying that if to their Number was added* 5, *and from this Sum ſubſtracting* 3, *and adding* 16 *to the Remainder, and from that Sum ſubſtracting* 50, *and adding* 54 *to the Remainder, he ſhould then have* 43 *Shillings. How many Shillings had he?*

Let $a =$ the Number of Shillings ſought, $b = 5$, $d = 3$, $m = 16$, $n = 50$, $p = 54$, $q = 43$.

The Perſon had a certain Number of Shillings, which call - -	1	a
To which 5 or b being added, we have - -	2	$a + b$
From which Sum ſubſtracting 3 or d, we have -	3	$a + b - d$
To which Remainder adding 16 or m, we have -	4	$a + b - d + m$
From which Sum ſubſtracting 50 or n, we have -	5	$a + b - d + m - n$
To which Remainder adding 54 or p, we have -	6	$a + b - d + m - n + p$
Which $a + b - d + m - n + p$ is to be equal to 43 or q, hence - -	7	$a + b - d + m - n + p = q$
The Queſtion being now expreſſed in *Algebra*, by tranſpoſing p, we have -	8	$a + b - d + m - n = q - p$

By

By transposing n we have -	9	$a+b-d+m=q-p+n$
By transposing m we have -	10	$a+b-d=q-p+n-m$
By transposing d we have -	11	$a+b=q-p+n-m+d$
Lastly, by transposing b we have	12	$a=q-p+n-m+d-b$

43 is the Number represented by q
—54 from which substracting 54 or p, there remains —11
—11 or $q-p$, see below.*
50 adding 50 or n to —11, the Sum is 39
39 or $q-p+n$
16 from which substracting 16 or m
23 or $q-p+n-m$
3 to which adding 3 or d
26 or $q-p+n-m+d$
5 from which substracting 5 or b
21 hence 21 is the Number sought; which is thus proved:

I say the Person had - -	21 Shillings
For if to them you add - -	5
	26
And from the Sum substract - -	3
	23
And to the Remainder add - -	16
	39
And from the Sum substract - -	50
There remains a *negative* or -	—11
And if to this Remainder we add -	54
It makes what the Question requires -	43

* When a *negative* Number is to be substracted from an *affirmative* Number, and the negative Number is greatest, as in this Case, it is only to take the Difference of the two Numbers, and place the Sign — before it; and if the next Number to be added is *affirmative*, and greater than the *negative* Remainder, then it is only substracting the *negative* Remainder from the *affirmative* Number which is to be added, and this will be the Sum.

If the Learner finds any Difficulty in conceiving this, he may collect all the *affirmative* Numbers into one Sum, and all the *negative* Numbers into another; and subtracting the Sum of the *Negatives* from the Sum of the *Affirmatives*, the Remainder is the Answer to the Question.

In the last Question,

The *affirmative* Quantities or Numbers are - - $q = 43$

$n = 50$

$d = 3$

96

Sum of the negative Numbers -75

$21 = a$ as before

The *negative* Quantities or Numbers are - $-p = -54$

$-m = -16$

$-b = -5$

-75

To reduce an Equation by Multiplication.

47. In the last Article, the unknown Quantity was connected with the known Quantities by the Signs $+$ or $-$ only, but it may happen that the unknown Quantity may be *divided* by some known Quantity; in this Case, *multiply every Part or all the Terms of the Equation by that known Quantity*; and the Part of the Equation containing the unknown Quantity will be then multiplied and divided by the same Quantity, take down this Equation, rejecting the known Quantity from that Part of the Equation where it both multiplies and divides the unknown Quantity, by Art. 20. it being in both Dividend and Divisor: After this Equation is set down, if there are any other Quantities connected with the unknown one by the Signs $+$ or $-$, transpose them to the other Side of the Equation as in the last Article, by which Method we shall have all the known Quantities on one Side of the Equation, and the unknown one by itself on the other Side, which is the Solution of the Question.

Question

Queſtion 7. *A Gameſter challenging another to play for as many Guineas as he had in his Hand, the other required to know how many there were, he replied, if you divide their Number by 5, and add 19 to the Quotient, I ſhall then have 23 Guineas in my Hand. How many Guineas had he?*

Let $a =$ the Number of Guineas ſought, $b = 5$, $d = 19$, $m = 23$.

Then the Gameſter had a certain Number of Guineas, which call -	1	a
Which being divided by 5 or b, the Quotient is by Art. 27. - -	2	$\frac{a}{b}$
To which Quotient $\frac{a}{b}$ if we add 19 or d, we have by Art. 6. - -	3	$\frac{a}{b} + d$
And this $\frac{a}{b} + d$, is to be equal to 23 or m, therefore we have - -	4	$\frac{a}{b} + d = m$
The Queſtion being expreſſed in *Algebra* by the Equation $\frac{a}{b} + d = m$, in which the unknown Quantity a being divided by b; now by the Rule, multiply every Part or Quantity in the Equation by b, and in this Multiplication, multiply only the Numerator a of the Quantity $\frac{a}{b}$ by b, according to the Rule of Vulgar Fractions in Arithmetic, and we have - - -	5	$\frac{ab}{b} + bd = bm$
Becauſe b is in both Dividend and Diviſor of the Quantity $\frac{ab}{b}$, hence by the Rule, rejecting b from $\frac{ab}{b}$ only, and placing down the remaining Part a, and all the other Parts of the Equation, without any Alteration, we have	6	$a + bd = bm$
Tranſpoſing bd by the laſt Article, it being a known Quantity, then -	7	$a = bm - bd$

Here the Queſtion is anſwered, for a the unknown Quantity is equal to the Product of the two Numbers repreſented by b and m, ſubſtracting from it the Product of the two Numbers repreſented by b and d.

The Number repreſented by b is 5, the Number repreſented by m is 23, which two Numbers being multiplied is bm or - - - - -	115
The Number repreſented by b is 5, the Number repreſented by d is 19, which two Numbers being multiplied is bd or - - - -	95
Subſtracting bd from bm, that is, 95 from 115, leaves $bm - bd$ or - - - - - -	20

Which is the Number ſought, or the Guineas the Gameſter had, and is proved from the Conditions of the Queſtion, thus,

I ſay the Gameſter had - -	20 Guineas
For if that Number is divided by 5, the Quotient is	4
But if to this 4 we add - - -	19
It makes what the Queſtion requires -	23

Queſtion 8. *To find that Number which being divided by* 15, *if to the Quotient we add* 27, *and ſubſtract* 13 *from the Sum, the Remainder will be* 18.

Let $a =$ the Number ſought, $b = 15$, $d = 27$, $m = 13$, $p = 18$.

Now I am to find a Number, which call - -	1	a
Which being divided by 15 or b, we have by Art. 27. -	2	$\frac{a}{b}$
To the Quotient or $\frac{a}{b}$, if we add 27 or d, we have by Art. 6. - -	3	$\frac{a}{b} + d$
From this Sum if we ſubſtract 13 or m, that is, connect m by the Sign $-$, it is -	4	$\frac{a}{b} + d - m$
Which $\frac{a}{b} + d - m$ is by the Queſtion to be equal to 18 or p, hence we have -	5	$\frac{a}{b} + d - m = p$

The

The Queſtion being now expreſſed in *Algebra* by this Equation $\frac{a}{b}+d-m=p$, and the unknown Quantity a being divided by b, multiply every Part of the Equation by b as in the laſt Queſtion, and then we have - - -	6	$\frac{ab}{b}+db-mb=pb$
Becauſe b is in both Dividend and Diviſor of the Quantity $\frac{ab}{b}$, reject b from this Quantity only as in the laſt Queſtion, placing down a and the remaining Quantities in the Equation without any Alteration, then we have - -	7	$a+db-mb=pb$
Becauſe mb is a known Quantity, tranſpoſe it by the Directions in the laſt Article, and we have - -	8	$a+db=pb+mb$
Becauſe db is a known Quantity, tranſpoſe it by the ſame Directions, and we have -	9	$a=pb+mb-db$

Now a the unknown Quantity being by itſelf on one Side of the Equation, the Queſtion is ſolved; for a, the unknown Quantity, is equal to the Product of the two Numbers repreſented by p and b, added to the Product of the two Numbers repreſented by m and b, ſubſtracting from this Sum the Product of the Numbers repreſented by the Letters d and b.

The Number repreſented by p is 18, the Number repreſented by b is 15, the Product of theſe two Numbers is pb or - - - -	270
The Number repreſented by m is 13, the Number repreſented by b is 15, the Product of theſe two Numbers is mb or - - - -	195
The Sum is $pb+mb$ or - - -	405

The

The Number represented by d is 27, the Number represented by b is 15, the Product of these two Numbers is 405, which being substracted from the Sum of the other two Products - - -	405
Leaves $pb + mb - db$ or a - -	60

Therefore 60 is equal to a, or 60 is the Number sought, which is thus proved from the Question.

I say the Number sought is - -	60
For if that is divided by 15 the Quotient is -	4
To which Quotient, or 4, if we add -	27
The Sum is - - - -	31
And if from this Sum we substract - -	13
There remains what the Question requires -	18

Question 9. *A Man being asked how many Shillings he had, replied, if you divide the Number I have by* 25, *and substract* 3 *from the Quotient, and then add* 51 *to this Remainder, and from the Sum substracting* 40, *I shall have* 12 *Shillings left. How many Shillings had he?*

Let $a =$ the Number of Shillings the Man had, $b = 25$, $d = 3$, $m = 51$, $p = 40$, $z = 12$.

Now the Man had a certain Number of Shillings, which call - -	1	a
Which being divided by 25 or b, we have by Art. 27. -	2	$\frac{a}{b}$
From the Quotient or $\frac{a}{b}$, if we substract 3 or d, that is, connecting d by the Sign $-$	3	$\frac{a}{b} - d$
To the Remainder adding 51 or m, we have by Art. 6. -	4	$\frac{a}{b} - d + m$
From which substracting 40 or p, that is, connecting p by the Sign $-$, we have -	5	$\frac{a}{b} - d + m - p$
Which $\frac{a}{b} - d + m - p$ is by the Question, to be equal to 12 or z, hence we have -	6	$\frac{a}{b} - d + m - p = z$

The

The Question being now expressed in *Algebra*, and the unknown Quantity a being divided by b, multiply every Quantity in the Equation by b, as in the two last Questions, then we have - -	7	$\frac{ab}{b} - db + mb - pb = zb$
And rejecting b out of the Quantity $\frac{ab}{b}$ only, because it is in both Dividend and Divisor, and setting down the rest as in the two last Questions, we have - -	8	$a - db + mb - pb = zb$
Because pb is a known Quantity, transpose it by Art. 46. and we have - -	9	$a - db + mb = zb + pb$
Because mb is a known Quantity, transpose it in like Manner, then we have -	10	$a - db = zb + pb - mb$
Because db is a known Quantity, by transposing it we have - -	11	$a = zb + pb - mb + db$

Now it appears the unknown Quantity, or a, is equal to the Product of the two Numbers represented by z and b, added to the Product of the two Numbers represented by p and b, substracting from this Sum the Product of the two Numbers represented by m and b, and adding to this Remainder the Product of the two Numbers represented by d and b.

The Number represented by z is 12, and that by b is 25, the Product of these two is zb or -	300
The Number represented by p is 40, and that by b is 25, the Product of these two is pb or -	1000
The Sum is $zb + pb$ or - -	1300
The Number represented by m is 51, and that by b is 25, the Product of these two is mb or -	1275
Which being substracted from the Sum of the other two, leaves $zb + pb - mb$ or - -	25
The Number represented by d is 3, and that by b is 25, the Product of these two is db or -	75
Which added to the last Remainder, the Sum is $zb + pb - mb + db$ or a	100

Whence

Whence the unknown Quantity a, or the Number of Shillings the Man had is 100, which is thus proved, from the Conditions of the Queſtion.

	Shillings.
I ſay the Man had - - -	100
For if that Number is divided by 25, the Quotient is	4
From which Quotient if we ſubſtract - -	3
Remains - - - - -	1
To which adding - - -	51
The Sum is - - - -	52
From which ſubſtracting - -	40
There remains what the Queſtion requires -	12

Queſtion 10. *A Country Servant, who underſtood* Algebra, *being aſked by his Maſter how many Cows there were in the Field, replied, if you add* 13 *to their Number, and divide that Sum by* 8, *and then add* 19 *to the Quotient, and ſubſtract* 11 *from that Sum, there will be* 12 *Cows left. How many Cows were there?*

Let $a=$ the Number of Cows, $b=13$, $d=8$, $m=19$, $p=11$, $x=12$.

Now there were in the Field a certain Number of Cows, which call - -	1	a
To which 13 or b being added, we have by Art. 6. -	2	$a+b$
Which $a+b$ being divided by 8 or d, we have by Art. 28.	3	$\frac{a+b}{d}$
To which if we add 19 or m, we have by Art. 6. - -	4	$\frac{a+b}{d}+m$
From which if we ſubſtract 11 or p, we have by connecting p with the Sign $-$ -	5	$\frac{a+b}{d}+m-p$
Which $\frac{a+b}{d}+m-p$ is by the Queſtion to be equal to 12 or x, hence we have -	6	$\frac{a+b}{d}+m-p=x$

Becauſe

Becaufe a, the unknown Term, is Part of the Fraction $\frac{a+b}{d}$ where the Divifor is d, therefore multiplying every Quantity in the Numerator of the Fraction by d, by the Rule of Vulgar Fractions in Arithmetick, and the other Quantities as before, then	7	$\frac{ad+bd}{d}+md-pd=xd$
Becaufe d is in every Term of the Dividend and Divifor of the Fraction $\frac{ad+bd}{d}$, reject d, from $\frac{ad+bd}{d}$ only, by Art. 22. and 24, and fet down all the reft as before, then - - -	8	$a+b+md-pd=xd$
Now begin to tranfpofe pd, it being a known Quantity, then we have	9	$a+b+md=xd+pd$
Becaufe md is a known Quantity, therefore tranfpofe it, and we have -	10	$a+b=xd+pd-md$
Becaufe b is a known Quantity, therefore tranfpofe it, and we have -	11	$a=xd+pd-md-b$

By this it appears that a, the unknown Quantity, is equal to the Product of the two Numbers reprefented by x and d, added to the Product of the two Numbers reprefented by p and d, fubftracting from this Sum the Product of the two Numbers reprefented by m and d, fubftracting ftill from this Remainder the Number reprefented by b.

The Product of the two Numbers reprefented by x and d is xd, or — — — —	96
The Product of the two Numbers reprefented by p and d is pd, or — — — —	88
Their Sum is $xd+pd$, or — —	184

The Product of the two Numbers represented by m and d is $m\,d$, or — — — —	152
Which subtracted from the Sum of the other two Products, there remains $x\,d + p\,d - m\,d$ —	32
From which subtracting the Number represented by b	13
The Remainder is $x\,d + p\,d - m\,d - b$, which is equal to a, or the Number sought — —	19

And that 19 Cows were in the Field, is thus proved from the Conditions of the Question.

I say the Number of Cows were — —	19
For if to them we add — — —	13
The Sum is — — —	32
Which divided by 8, the Quotient is —	4
To which Quotient if we add — —	19
The Sum is — — —	23
From which Sum subtracting — —	11
There remains what the Question requires —	12

Question 11. *Two young Gentlemen were disputing how many Men were at a public Diversion, but not agreeing, they referred it to a third Person, who, being skilled in* Algebra, *instead of a direct Answer, replied, that if you subtract* 115 *from their Number, and divide the Remainder by* 50, *and add* 39 *to that Quotient, from which Sum subtracting* 16, *and adding* 68 *to the Remainder, this last Sum will be equal to* 101. *How many Men were there?*

Let $a =$ the Number of Men sought, $b = 115$, $c = 50$, $d = 39$, $n = 16$, $p = 68$, $x = 101$.

There were a certain Number of Men, which call - -	1	a
From which 115, or b, being subtracted, we have	2	$a - b$
Which Remainder of $a - b$, being divided by 50, or c, we have by Art. 28.	3	$\frac{a-b}{c}$
To which Quotient if we add 39, or d, we have	4	$\frac{a-b}{c} + d$

From

From this Sum if we ſubſtract 16 or n, we have	5	$\frac{a-b}{c}+d-n$
To which Remainder if we add 68 or p, we have	6	$\frac{a-b}{c}+d-n+p$
Which $\frac{a-b}{c}+d-n+p$ is, by the Queſtion, to be equal to 101 or x, hence	7	$\frac{a-b}{c}+d-n+p=x$
Becauſe a, the unknown Quantity, is Part of $\frac{a-b}{c}$, which being divided by c, therefore multiplying by c, as in the laſt Queſtion, we have	8	$\frac{ca-cb}{c}+cd-cn+cp=cx$
Becauſe c is in every Term of the Dividend and Diviſor of $\frac{ca-cb}{c}$ reject c, as in the laſt Queſtion, and ſet down all the reſt as before, and we have	9	$a-b+cd-cn+cp=cx$
Now tranſpoſe cp, it being a known Quantity, then it is	10	$a-b+cd-cn=cx-cp$
Tranſpoſe cn, it being a known Quantity, and we have	11	$a-b+cd=cx-cp+cn$
Tranſpoſe cd, it being a known Quantity, and we have	12	$a-b=cx-cp+cn-cd$
And tranſpoſing b, it being a known Quantity, we have	13	$a=cx-cp+cn-cd+b$

Hence it appears that a, the unknown Quantity, is equal to the Product of the two Numbers repreſented by c and x, ſubſtracting from it the Product of the two Numbers repreſented by c and p,

O 2 adding

adding to that Remainder the Product of the two Numbers represented by c and n, substracting from this Sum the Product of the two Numbers represented by c and d, and adding to this Remainder the Number represented by b.

The Product of the two Numbers represented by c and x is cx, or - - - -	5050
The Product of the two Numbers represented by c and p is cp, or - - -	3400
The Remainder is $cx - cp$, or - -	1650
The Product of the two Numbers represented by c and n is cn, or - - -	800
Which added to the last Remainder, the Sum is $cx - cp + cn$, or - - -	2450
The Product of the two Numbers represented by c and d is cd, or 1950, which being substracted -	1950
The Remainder is $cx - cp + cn - cd$, or -	500
Adding the Number represented by b, or -	115
The Sum is $cx - cp + cn - cd + b$, which is equal to a, the Number sought - - -	615

And that there were 615 Men is proved from the Conditions of the Question.

	Men.
I say there were - - -	615
For if from them we substract - -	115
Remains - - - - -	500
Which being divided by 50, the Quotient is -	10
To which adding - - -	39
The Sum is - - -	49
From which substracting - - -	16
The Remainder is - - - -	33
To which adding - - -	68
There remains what the Question requires -	101

Question 12. *There is a certain Number to which* 9 *being added, and dividing this Sum by* 5, *if from this Quotient we substract* 6, *and add* 101 *to the Remainder, from that Sum substracting* 10, *there remains* 97. *What is the Number?*

Let

Let $a =$ the Number sought, $b = 9$, $c = 5$, $d = 6$, $m = 101$, $p = 10$, $x = 97$.

Now I am to find a certain Number, which call -	1	a
To which 9, or b, being added, we have by Art. 6.	2	$a + b$
This being divided by 5, or c, we have by Article 28. - -	3	$\dfrac{a+b}{c}$
From which substracting 6, or d, we have -	4	$\dfrac{a+b}{c} - d$
To which adding 101, or m, we have by Art. 6.	5	$\dfrac{a+b}{c} - d + m$
From this substracting 10, or p, that is, connecting p by the Sign —, it is	6	$\dfrac{a+b}{c} - d + m - p$
Which $\dfrac{a+b}{c} - d + m - p$ is to be equal to 97, or x, hence - -	7	$\dfrac{a+b}{c} - d + m - p = x$
The Question being thus expressed in *Algebra*, begin and multiply by c, for the Reason in the last Question, then we have	8	$\dfrac{ca+cb}{c} - cd + cm - cp = cx$
Rejecting c from $\dfrac{ca+cb}{c}$, and setting down the rest as before, then -	9	$a + b - cd + cm - cp = cx$
Now transposing $c\,p$, it being a known Quantity, we have - -	10	$a + b - cd + cm = cx + cp$
Transposing cm, it being a known Quantity, we have - -	11	$a + b - cd = cx + cp - cm$
Transposing $c\,d$, it being a known Quantity, we have - -	12	$a + b = cx + cp - cm + cd$
Lastly, transposing b, it being a known Quantity, we have - -	13	$a = cx + cp - cm + cd - b$

The *Algebraick* Operation being finiſhed, the *Numerical* Work is thus.

The Product of the two Numbers repreſented by c and x is cx, or	485
The Product of the two Numbers repreſented by c and p is cp, or	50
The Sum is $cx + cp$, or - -	535
The Product of the two Numbers repreſented by c and m is cm, or - - - -	505
Subſtracting, the Remainder is $cx + cp - cm$, or -	30
The Product of the two Numbers repreſented by c and d is cd, or - - - -	30
Adding, the Sum is $cx + cp - cm + cd$, or -	60
The Number repreſented by b is 9, ſubſtracting -	9
The Remainder is $cx + cp - cm + cd - b$, which is equal to a, or the Number ſought - -	51

And is thus proved from the Conditions of the Queſtion.

I ſay the Number ſought is - -	51
For if to this we add - - -	9
The Sum is - - -	60
Which being divided by 5, the Quotient is -	12
From which ſubſtracting - - -	6
The Remainder is - - -	6
To which adding - - - -	101
The Sum is - - - -	107
From which ſubſtracting - - -	10
There remains what the Queſtion requires -	97

To reduce an Equation by Diviſion.

48. In the laſt Article the unknown Quantity was divided by a known Quantity, in the Equation that aroſe from the Conditions of the Queſtion; in this Article the unknown Quantity will be *multiplied* into a known Quantity, in the Equation that ariſes from the Conditions of the Queſtion; when this happens, *divide every Quantity on both Sides of the Equation, by the ſame known Quantity into which the unknown Quantity is multiplied,* then

then you will find the unknown Quantity to be multiplied and divided by the ſame Quantity; now place down this Equation, rejecting only the Letter from that Quantity, where it multiplies and divides the unknown Quantity, as in the laſt Article; then tranſpoſe the Quantities as before, but if there are none to be tranſpoſed the Queſtion is ſolved.

If any Quantities are connected with the unknown one by the Signs + or —, it will be moſt convenient for the Learner to tranſpoſe them before he begins to divide by the Rule juſt given.

Queſtion 13. *A Perſon required another to tell him how many Shillings he had, by ſaying that if their Number was multiplied by* 13, *and if from that Product was ſubſtracted* 25, *he ſhould then have* 170 *Shillings. How many Shillings had he?*

Let $a =$ the Number of Shillings the Perſon had, $b = 13$, $d = 25$, $m = 170$.

A Perſon had a certain Number of Shillings, which call -	1	a
Which multiplied by 13, or b, we have by Art. 9. - - -	2	$b\,a$
From the Product, or $b\,a$, if we ſubſtract 25, or d, we have -	3	$b\,a - d$
Which Remainder $b\,a - d$ is by the Queſtion to be equal to 170, or m, hence - - - - -	4	$b\,a - d = m$
Becauſe d is on the ſame Side of the Equation with the unknown Quantity, and connected by the Sign —, therefore tranſpoſe d, then -	5	$b\,a = m + d$
There being no more Quantities to be tranſpoſed, and the unknown Quantity being multiplied by b, therefore divide both Sides of the Equation by b. Now $b\,a$ divided by b gives $\frac{b\,a}{b}$ by Art. 27. and $m + d$ divided by b, gives $\frac{m+d}{b}$ by Art. 28. therefore we have	6	$\frac{b\,a}{b} = \frac{m+d}{b}$

Becauſe

Becauſe b is in both Dividend and Diviſor of the Quantity $\frac{ba}{b}$, reject b by Art. 20. and putting down the other Quantities without any Alteration as in the foregoing Queſtion, we have - - -	7	$a = \frac{m+d}{b}$

From hence it appears that a, the unknown Quantity, is equal to the Sum of the two Numbers repreſented by m and d, divided by the Number repreſented by b.

The Number repreſented by m is - -	170
The Number repreſented by d is - -	25
The Sum is $m + d$, or - - -	195

And dividing 195, or $m + d$, by 13, or b, the Quotient is $\frac{m+d}{b}$, or 15, which is a, or the Number ſought.

The Truth of which is thus proved from the Conditions of the Queſtion.

I ſay the Perſon had - -	15 Shillings
For if that is multiplied by - -	13
	45
	15
The Product is - - -	195
From which ſubſtracting - -	25
There remains what the Queſtion requires -	170

Queſtion 14. *A Butcher ſeeing a Drover going to Market with a Number of Sheep, aſked how many there were; the Drover anſwered, if you multiply their Number by* 9, *and ſubſtract* 157 *from that Product, and add* 168 *to the Remainder, I ſhall then have* 2000 *Sheep. How many Sheep had he?*

Let $a =$ the Number of Sheep, $b = 9$, $d = 157$, $m = 168$, $p = 2000$.

Then

The Drover had a certain Number of Sheep, which call -	1	a
Which multiplied by 9, or b, we have by Art. 9. - -	2	$b\,a$
From the Product ſubſtracting 157, or d, that is, connecting d by the Sign —, we have - -	3	$b\,a - d$
To which Remainder adding 168, or m, we have by Art. 6. -	4	$b\,a - d + m$
This $b\,a - d + m$ is by the Queſtion to be equal to 2000, or p, hence we have - -	5	$b\,a - d + m = p$
Now according to the Rule begin with tranſpoſing m, and we have -	6	$b\,a - d = p - m$
Then tranſpoſing d, we have -	7	$b\,a = p - m + d$
The Quantities being all tranſpoſed that were connected by the Signs + or —, and the unknown Quantity being multiplied by b, therefore by the Rule divide both Sides of the Equation by b, but $b\,a$ divided by b, gives $\frac{b\,a}{b}$, and $p - m + d$ divided by b, gives $\frac{p - m + d}{b}$, by Art. 28. hence we have - -	8	$\frac{b\,a}{b} = \frac{p - m + d}{b}$
Rejecting b from the Quantity $\frac{b\,a}{b}$, becauſe it is in both Dividend and Diviſor, and placing down the remaining Parts of the Equation without any Alteration as before, we have	9	$a = \frac{p - m + d}{b}$

The *Algebraic* Work is now finiſhed, for the unknown Quantity a is on one Side of the Equation by itſelf, and it appears to be equal to the Number repreſented by p, ſubſtracting from it the Number repreſented by m, adding to this Remainder the Number repreſented by d, and dividing this Sum by the Number repreſented by b.

The Number represented by p is - -	2000
From which substracting the Number represented by m, which is - - -	168
There remains $p - m$, or - -	1832
To which adding the Number represented by d -	157
The Sum is $p - m + d$, or - -	1989

And dividing this 1989, or $p - m + d$, by 9, the Number represented by b, the Quotient is $\frac{p - m + d}{b}$, or 221, which is a, or the Number of Sheep the Drover had; and is proved by the Conditions of the Question thus.

I say the Number of Sheep were - -	221
For that being multiplied by - -	9
The Product is - - -	1989
From which substracting - -	157
There remains - - -	1832
To which adding - - -	168
The Sum is what the Question requires -	2000

Question 15. *A Man being asked what he gave for his Horse, answered, if you multiply the Number of Pounds I gave by* 5, *and then add* 15 *to the Product, and from that Sum substract* 50, *and to the Remainder adding* 25, *from which Sum substracting* 14, *this Remainder will be equal to* 81. *What did he give for his Horse?*

Let $a =$ what he gave for the Horse, $b = 5$, $d = 15$, $c = 50$, $p = 25$, $m = 14$, $x = 81$.

Let the Pounds which the Person gave for the Horse be called	1	a
Which multiplied by 5, or b, we have by Art. 9. -	2	ba
To which Product if we add 15, or d, we have by Art. 6.	3	$ba + d$
From this Sum substracting 50, or c, that is, connecting c by the Sign —, we have -	4	$ba + d - c$

To

To which Remainder adding 25, or p, we have -	5	$ba+d-c+p$
From which Sum ſubſtracting 14, or m, we have -	6	$ba+d-c+p-m$
Which $ba+d-c+p-m$ is by the Queſtion to be equal to 81, or x, hence we have - -	7	$ba+d-c+p-m=x$
Now tranſpoſing m, we have	8	$ba+d-c+p=x+m$
And tranſpoſing p, we have	9	$ba+d-c=x+m-p$
And tranſpoſing c, we have	10	$ba+d=x+m-p+c$
And tranſpoſing d, we have	11	$ba=x+m-p+c-d$
The Quantities connected by the Signs $+$ or $-$, being now all tranſpoſed, I obſerve the unknown Quantity to be multiplied by b, therefore divide every Term on both Sides of the Equation by b. Now dividing ba by b, it is $\frac{ba}{b}$, and dividing $x+m-p+c-d$ by b, we have $\frac{x+m-p+c-d}{b}$ as in the foregoing Queſtions, hence we have -	12	$\frac{ba}{b}=\frac{x+m-p+c-d}{b}$
Rejecting b from the Quantity $\frac{ba}{b}$, becauſe it is in both Dividend and Diviſor, and placing down the reſt of the Equation without any Alteration as before, and we have - -	13	$a=\frac{x+m-p+c-d}{b}$

That is, a, the unknown Quantity, is equal to the Number repreſented by x, added to the Number repreſented by m, ſubſtracting from their Sum the Number repreſented by p, adding to this Remainder the Number repreſented by c, ſubſtracting from this Sum the Number repreſented by d, and dividing this Remainder by the Number repreſented by b.

Now x is - - - -	81
To which adding m, or - - -	14
The Sum is $x+m$, or - - -	95
From which fubftracting p, or - -	25
There remains $x+m-p$, or - -	70
To which adding c, or - - -	50
The Sum is $x+m-p+c$, or - -	120
From which fubftracting d, or - - -	15
There remains $x+m-p+c-d$, or -	105

Now dividing this 105, or $x+m-p+c-d$, by b, or 5, the Quotient is $\frac{x+m-p+c-d}{b}$, or 21, which is equal to a, or Number of Pounds the Horfe coft.

Which is proved from the Conditions of the Queftion, thus,

I fay the Horfe coft - -	21 Pounds
For if that is multiplied by - -	5
The Product is - -	105
To which adding - - -	15
The Sum is - - -	120
From which fubftracting - -	50
There remains - - -	70
To which adding - - -	25
The Sum is - - -	95
From which fubftracting - - -	14
There remains what the Queftion requires -	81

Queftion 16. *There is a certain Number which being multiplied by* 7, *if from this Product we fubftract* 21, *and to the Remainder add* 11, *and from this Sum fubftract* 23, *and add to the Remainder* 33, *this laft Sum will be* 210. *What is the Number?*

Let $a=$ the Number fought, $b=7$, $d=21$, $x=11$, $c=23$, $p=33$, $r=210$.

Now

Now there is a certain Number fought, which call	1	a
Which multiplied by 7, or b, we have by Art. 9.	2	ba
From which fubftracting 21, or d, that is, connecting d by the Sign —, we have	3	$ba-d$
To this adding 11, or x, we have by Art. 6.	4	$ba-d+x$
From which fubftracting 23, or c, that is, connecting c by the Sign —, we have	5	$ba-d+x-c$
To which adding 33, or p, we have by Art. 6.	6	$ba-d+x-c+p$
And this $ba-d+x-c+p$ is by the Queftion to be equal to 210, or r, hence we have	7	$ba-d+x-c+p=r$
The Queftion being now expreffed in *Algebra*, begin the Solution by tranfpofing p, and then we have	8	$ba-d+x-c=r-p$
Tranfpofing c we have	9	$ba-d+x=r-p+c$
Tranfpofing x we have	10	$ba-d=r-p+c-x$
Tranfpofing d we have	11	$ba=r-p+c-x+d$
All the Quantities being now tranfpofed that were connected by the Signs + or —, and the unknown Quantity being multiplied by b, divide every Term, or both Sides of the Equation by b, as in the laft Example, and we have	12	$\frac{ba}{b}=\frac{r-p+c-x+d}{b}$
Now reject b out of the Quantity $\frac{ba}{b}$, becaufe it is in both Dividend and Divifor, and fetting down the remaining Parts of the Equation, as in the laft Queftion, and we have	13	$a=\frac{r-p+c-x+d}{b}$

To

To find what a is in Numbers.

The Number represented by r, is - -	210
From which subtracting the Number represented by p, which is - - -	33
There remains $r-p$, or - - -	177
To which adding the Number represented by c -	23
The Sum is $r-p+c$, or - -	200
From which subtracting the Number represented by x	11
There remains $r-p+c-x$, or - -	189
To which adding the Number represented by d	21
The Sum is $r-p+c-x+d$, or -	210

And dividing this 210 by b, or 7, the Quotient is $\frac{r-p+c-x+d}{b}$, or 30, which is equal to a, or the Number sought, and is thus proved.

I say the Number sought is - -	30
For if this is multiplied by - -	7
The Product is - - -	210
From which subtracting - -	21
There remains - - - -	189
To which adding - - -	11
The Sum is - - - -	200
From which subtracting - -	23
There remains - - -	177
To which adding - - -	33
The Sum is what the Question requires -	210

Question 17. *A Gamester challenged another to play with him for as many Guineas as were in his Hand; but being asked how many they were, answered, if you multiply their Number by* 10, *and substract* 100 *from the Product, and to the Remainder add* 55, *and from the Sum substract* 31, *and adding to this Remainder* 115, *I shall then have* 539 *Guineas. How many had he?*

Let $a=$ the Number of Guineas sought, $b=10$, $c=100$, $d=55$, $m=31$, $x=115$, $p=539$.

Then

Then a Gamester had a certain Number of Guineas, which call - -	1	a
Which being multiplied by 10, or b, we have by Art. 9.	2	ba
From which subtracting 100, or c, we have -	3	$ba - c$
To which adding 55, or d, we have by Art. 6. -	4	$ba - c + d$
From this subtracting 31, or m, we have -	5	$ba - c + d - m$
To which adding 115, or x, we have - -	6	$ba - c + d - m + x$
This by the Question is to be equal to 539, or p, hence we have - -	7	$ba - c + d - m + x = p$
Then by transposing x we have	8	$ba - c + d - m = p - x$
Transposing m it is -	9	$ba - c + d = p - x + m$
Transposing d we have -	10	$ba - c = p - x + m - d$
And transposing c -	11	$ba = p - x + m - d + c$
Now divide by b, as before directed, and we have -	12	$\frac{ba}{b} = \frac{p - x + m - d + c}{b}$
And rejecting b from $\frac{ba}{b}$, and placing down the rest as before, then - -	13	$a = \frac{p - x + m - d + c}{b}$

The Question being now solved in *Algebra*, we are to find what a is equal to in Numbers.

Now p is equal to - - -	539
From which subtracting x, or - -	115
There remains $p - x$, or - - -	424
To this adding m, or - - -	31
The Sum is $p - x + m$, or - - -	455
From this subtracting d, or - -	55
There remains $p - x + m - d$, or -	400
To this adding c, or - - -	100
The Sum is $p - x + m - d + c$, or -	500

But dividing this 500 by b, which is 10, the Quotient is 50, the Number of Guineas the Gameſter had; and is thus proved from the Conditions of the Queſtion.

I ſay the Gameſter had - -	50 Guineas
For if that Number is multiplied by -	10
The Product is - - -	500
From which ſubſtracting - -	100
There remains - - -	400
To which adding - - -	55
The Sum is - - -	455
From which ſubſtracting - -	31
There remains - - -	424
To which adding - -	115
The Sum is what the Queſtion requires -	539

Queſtion 18. *A Perſon being aſked how many Hours it was paſt Noon, replied, if you multiply the Number of Hours paſt Noon by* 7, *and ſubſtract* 5 *from the Product, and to the Remainder add* 9, *and from the Sum ſubſtract* 3, *and to this Remainder adding* 4, *this Sum will be equal to* 12. *How many Hours was it paſt Noon, or what of the Clock was it?*

Let $a =$ the Number of Hours it was paſt Noon, or the Number ſought, $m = 7$, $p = 5$, $d = 9$, $c = 3$, $b = 4$, $x = 12$.

Then there is a certain Number of Hours paſt Noon, which call - - -	1	a
This multiplied by 7, or m, we have by Art. 9. -	2	ma
From which ſubſtracting 5, or p, we have - -	3	$ma - p$
To this adding 9, or d, we have by Art. 6. - -	4	$ma - p + d$
From which ſubſtracting 3, or c, that is, connecting c by the Sign —, we have -	5	$ma - p + d - c$
To which adding 4, or b, we have by Art. 6. -	6	$ma - p + d - c + b$

Which

Which by the Question is to be equal to 12, or x, hence	7	$ma-p+d-c+b=x$
Now transpose b, and we have	8	$ma-p+d-c=x-b$
Transposing c, then -	9	$ma-p+d=x-b+c$
Transposing d, and -	10	$ma-p=x-b+c-d$
Transposing p, we have -	11	$ma=x-b+c-d+p$
Now dividing by m, as in the former Questions, and we have - -	12	$\frac{ma}{m}=\frac{x-b+c-d+p}{m}$
Rejecting m from the Quantity $\frac{ma}{m}$, as before, and we have	13	$a=\frac{x-b+c-d+p}{m}$

The *Algebraic* Work being finished, we may find what a is in Numbers thus.

Now x is equal to - - -	12
From which substracting b, or - -	4
There remains $x-b$, or - -	8
To which adding c, or - -	3
The Sum is $x-b+c$, or -	11
From which substracting d, or - -	9
There remains $x-b+c-d$, or - -	2
To which adding p, or - - -	5
The Sum is $x-b+c-d+p$, or - -	7

And dividing this by m, or 7, the Quotient is 1, which is equal to a, or the Number of Hours it was past Noon, hence it was 1 of the Clock in the Afternoon.

Which is thus proved from the Conditions of the Question.

I say the Number of Hours past Noon were -	1
For if that is multiplied by - -	7
The Product is - - -	7
From which substracting - - -	5
There remains - - - -	2
To which adding - - -	9
The Sum is - - - -	11
From which substracting - - -	3
There remains - - -	8
To which adding - - -	4
The Sum is what the Question requires -	12

To reduce an Equation by Involution.

49. Hitherto there has been no Equation in which the unknown Quantity has had the radical Sign prefixt before it, or has been connected with known Quantities under the radical Sign; but as this is a Case which frequently happens, we are now to explain the Manner, how such Equations are managed.

If any Part of an Equation is a *surd* Quantity, but the unknown Quantity is not under the radical Sign, then there is no Occasion to clear this Equation of its *Surds*; but if the unknown Quantity is under the radical Sign, then the Equation must be cleared of its *Surds*.

And when there is a given Equation where the *unknown Quantity is under the radical Sign*, and there are known Quantities without the radical Sign on that Side of the Equation, and connected by the Signs + or —, transpose all those Quantities which are without the radical Sign, to the other Side of the Equation; then raise both Sides of the Equation to the Square, if the radical Sign expresses the Square Root, or to the Cube, if the radical Sign expresses the Cube Root, and so on; by which Means the Equation will be cleared of its *Surds*.

After this, if there are no known Quantities on the same Side of the Equation with the unknown one, the Question is solved; but if there are still known Quantities on the same Side of the Equation with the unknown Quantity, the Equation is to be reduced by some of the Methods before explained, at Art. 46, 47, 48.

The Square Root is expressed by this Sign $\sqrt{}$, and the Cube Root by the same Sign with a 3 on the Top, thus $\sqrt[3]{}$; and if any Root is taken besides the Square Root, the Figure over the Sign shews what Root it is; but when it is only the Square Root, then there is generally no Figure over the Sign.

Question 19. *Two Gentlemen were talking of the Number of Acres there were in a Park, the Park-Keeper being present, and disposed to show his Learning, told them, that if they extracted the square Root of the Number of Acres in the Park, from which square Root substracting* 5, *this Remainder will be equal to* 50. *How many Acres were there in the Park?*

Let

Let $a =$ the Number of Acres in the Park, $b = 5$, $d = 50$.

Now there were a certain Number of Acres in the Park, which call	1	a
The Square Root of which, by Art. 33. is - -	2	$\sqrt{a}$
From which substracting 5, or b, that is, connecting b by the Sign —, it is	3	$\sqrt{a}:-b$
Which $\sqrt{a}:-b$ by the Question is equal to 50, or d, hence -	4	$\sqrt{a}:-b=d$
The Question being now expressed in *Algebra*, and observing that b is not under the radical Sign, therefore transpose b, then -	5	$\sqrt{a}=d+b$
Now all the Quantities being transposed, which were not under the radical Sign, square both Sides of the Equation, as the radical Sign expresses the Square Root. But the Square of $\sqrt{a}$ is a, by Art. 43. and the Square of $d+b$ is $dd+2db+bb$, by Art. 32. and making these equal to one another, for the Square of equal Quantities or Numbers must be equal, and we have -	6	$a=dd+2db+bb$

Hence it appears that a, the unknown Quantity, is equal to the Square of the Number represented by d, added to twice the Product of the two Numbers represented by d and b, and this Sum added to the Square of the Number represented by b.

The Square of the Number represented by d is dd, or	2500
The Product of the two Numbers represented by d and b is db, or 250, and twice that Product is $2db$, or - - - -	500
The Sum is $dd+2db$, or - -	3000
The Square of the Number represented by b is bb, or	25
The Sum is $dd+2db+bb$, or 3025, which is $= a$, or the Number sought - - -	3025

Hence, I ſay, there were 3025 Acres in the Park, which is thus proved, from the Conditions of the Queſtion.

The Number of Acres in the Park were -	3025
Now the Square Root of that Number is -	55
From which ſubſtracting - - -	5
There remains what the Queſtion requires -	50

Queſtion 20. *A Perſon, who had been fortunate at Gaming, was aſked how many Guineas he had won, to which he anſwered, that if the Square Root of their Number was extracted, from which Root ſubſtracting 7, he ſhould then have 16 Guineas. What Number of Guineas did he win?*

Let $a =$ the Number of Guineas won, $b = 7$, $d = 16$.

Now a Perſon won a Number of Guineas, which call -	1	a
The ſquare Root of which by Art. 33. is - -	2	$\sqrt{a}$
From which ſubſtracting 7, or b, we have - -	3	$\sqrt{a} - b$
Which $\sqrt{a} - b$ by the Queſtion is to be equal to 16, or d, hence	4	$\sqrt{a} - b = d$
Now becauſe b is not under the radical Sign, therefore tranſpoſe b, then - -	5	$\sqrt{a} = d + b$
All the Quantities not under the radical Sign being now tranſpoſed, in order to clear the Equation of the Surd, raiſe both Sides of the Equation to the Square or ſecond Power. But the Square of $\sqrt{a}$ is a, by Art. 43. and the Square of $d + b$ is $dd + 2db + bb$, by Art. 32. and making theſe two equal to one another, for the Square of equal Quantities or Numbers muſt be equal, and we have	6	$a = dd + 2db + bb$

That is to ſay, the unknown Quantity, or a, is equal to the Square of the Number repreſented by d, added to twice the Product of the two Numbers repreſented by d and b, to which Sum add the Square of the Number repreſented by b.

The

The Square of the Number reprefented by d is dd, or	256
The Product of the two Numbers reprefented by d and b is db, or 112, and twice that Product is $2db$, or	224
The Sum is $dd + 2db$, or - -	480
The Square of the Number reprefented by b is bb, or	49
The Sum is $dd + 2db + bb$, or 529, which is equal to a, or the Number fought - -	529

Therefore the Perfon won 529 Guineas; and is thus proved, from the Conditions of the Queftion.

I fay the Number of Guineas he won was -	529
For the Square Root of that Number is -	23
And if from that Square Root we fubftract -	7
There remains what the Queftion requires -	16

Queftion 21. *A Gentleman having fold his Eftate, an impertinent illiterate Perfon afked him what he had fold it for, why, Sir, replied he, if you extract the Square Root of the Number of Guineas for which I fold it, and add 17 to that Number, this Sum will be equal to 317. How many Guineas had the Gentleman for his Eftate?*

Let $a =$ the Number of Guineas for which the Eftate was fold, $b = 17$, $d = 317$.

Now the Eftate was fold for a Number of Guineas, which call	1	a
The fquare Root of which by Art. 33. is - -	2	$\sqrt{a}$
To which 17, or b, being added, we have - - -	3	$\sqrt{a} + b$
Which $\sqrt{a} + b$ by the Queftion is to be equal to 317, or d, hence	4	$\sqrt{a} + b = d$
The Queftion being now expreffed in *Algebra*, and becaufe b is not under the radical Sign, therefore tranfpofe b, and we have - -	5	$\sqrt{a} = d - b$
Now fquare both Sides of the Equation, and make them equal to one another, for the Reafons mentioned in the two laft Queftions, and we have	6	$a = dd - 2db + bb$

From

From hence we know that a, the unknown Quantity, is equal to the Square of the Number repreſented by d, ſubſtracting from it twice the Product of the Numbers repreſented by d and b, and adding to the Remainder the Square of the Number repreſented by b.

The Square of the Number repreſented by d is dd, or	100489
The Product of the two Numbers repreſented by b and d is db, or 5389, and twice that Product is $2db$, or	10778
Which ſubſtracted, the Remainder is $dd - 2db$, or	89711
The Square of the Number repreſented by b is bb, or	289
The Sum is $dd - 2db + bb$, or 90000, which is equal to a, and is the Number ſought	90000

And that the Eſtate was ſold for 90000 Guineas, is thus proved from the Conditions of the Queſtion.

I ſay the Eſtate was ſold for	90000 Guineas
For the ſquare Root of that is	300
To which if we add	17
The Sum is what the Queſtion requires	317

Queſtion 22. *A young Gentleman, when he came of Age, aſked his Guardian the annual Rent of the Eſtate his Father left him, to which he was anſwered, that if he extracted the ſquare Root of the Number of Pounds for which the Eſtate was rented, and to this Root if he added 27, it would be equal to 100 Pounds. What was the annual Rent of the Eſtate?*

Let $a =$ the Rent of the Eſtate, $m = 27$, $x = 100$.

Now the Rent of the Eſtate was	1	a
The ſquare Root of which by Art. 33. is	2	$\sqrt{a}$
To which 27, or m, being added, we have	3	$\sqrt{a} + m$
Which by the Queſtion is to be equal to 100, or x, hence	4	$\sqrt{a} + m = x$

The

The Queſtion being now expreſſed in *Algebra*, begin by tranſpoſing m, for the Reaſons mentioned in the former Queſtions, and then we have - -	5	$\sqrt{a} = x - m$
Now ſquaring both Sides of the Equation, to take away the radical Sign, as was done in the foregoing Queſtions, and then we have -	6	$a = xx - 2xm + mm$

And there being no more Quantities to be tranſpoſed, the Queſtion is ſolved; for we may find the Value of a in Numbers from the *Algebraic* Work, thus:

The Square of the Number repreſented by x is xx, or	10000
The Product of the two Numbers repreſented by the Letters x and m is xm, or 2700, and twice that Product is $2xm$, or - -	5400
Which ſubſtracted leaves $xx - 2xm$, or -	4600
The Square of the Number repreſented by m is mm, or	729
The Sum is 5329, or $xx - 2xm + mm$, which is equal to a, or the Number ſought - -	5329

And that the annual Rent of the Eſtate was 5329 Pounds, is proved from the Conditions of the Queſtion.

I ſay the annual Rent of the Eſtate was -	5329 Pounds
For the ſquare Root of that is -	73
To which there being added - -	27
The Sum is what the Queſtion requires -	100

Queſtion 23. *To find that Number to which* 1290 *being added, if the ſquare Root of this Sum is extracted, from which Root ſubſtracting* 29, *the Remainder may be* 71.

Let $a =$ the Number ſought, $b = 1290$, $d = 29$, $x = 71$.

There is a Number ſought, which I call - -	1	a
To which 1290, or b, being added, we have - -	2	$a + b$
The ſquare Root of which Sum by Art. 34. is - -	3	$\sqrt{a + b}$

I From

From which ſubſtracting 29, or d, we have - -	4	$\sqrt{a+b}:-d$
Which by the Queſtion is equal to 71, or x, hence -	5	$\sqrt{a+b}:-d=x$
Now begin the Solution with tranſpoſing d, it not being under the radical Sign, and then - -	6	$\sqrt{a+b}=x+d$
All the Quantities on one Side of the Equation being now under the radical Sign, to take away that, as the unknown Quantity is under it, ſquare both Sides of the Equation as before. Now the Square of $\sqrt{a+b}$ is $a+b$, by Art. 43. and the Square of $x+d$ is $xx+2xd+dd$, by Art. 32. and as the Squares of equal Numbers, or Quantities, muſt be equal to one another, hence - -	7	$a+b=xx+2xd+dd$
Now tranſpoſe b, it being a known Quantity, and then	8	$a=xx+2xd+dd-b$

From whence we may find the Value of a in Numbers.

The Square of the Number repreſented by x is xx, or	5041
The Product of the two Numbers repreſented by x and d is xd, or 2059, and twice that Product is $2xd$, or - - - - -	4118
The Sum is $xx+2xd$, or - - -	9159
The Square of the Number repreſented by d is dd, or	841
The Sum is $xx+2xd+dd$, or - -	10000
From which ſubſtracting the Number repreſented by b	1290
There remains 8710, or $xx+2xd+dd-b$, which is $=a$, or the Number ſought -	8710

And is thus proved from the Conditions of the Queſtion.

I ſay

I say the Number sought is - - -	8710
For if to that we add - - -	1290
The Sum is - - - -	10000
The square Root of which is - -	100
From which substracting - -	29
There remains what the Question requires -	71

Question 24. *A Person being asked his Age, replied, that if from my Age you substract* 11, *and extract the square Root of the Remainder, to which Root adding* 13, *this Sum will be equal to* 20. *What was the Age of the Person?*

Let $a =$ the Number of Years, or Age of the Person, $b = 11$, $m = 13$, $d = 20$.

Now the Age of the Person is	1	a
From which if we substract 11, or b, we have -	2	$a - b$
The square Root of which by Art. 34. is - -	3	$\sqrt{a-b}$
To which adding 13, or m, we have - -	4	$\sqrt{a-b} : + m$
Which by the Question is equal to 20, or d; hence we have	5	$\sqrt{a-b} : + m = d$
The Question being thus expressed in *Algebra*, and m not being under the radical Sign, therefore transpose m; then	6	$\sqrt{a-b} = d - m$
Now square both Sides of the Equation, to clear it of the Surd, as in the former Questions. But the Square of $\sqrt{a-b}$, is $a-b$, by Art. 43. and the Square of $d-m$ by Art. 32. is $dd - 2dm + mm$; then as the Squares of equal Quantities are equal, we have - -	7	$a - b = dd - 2dm + mm$
And by transposing b we have	8	$a = dd - 2dm + mm + b$

By which we find what a is in Numbers. Thus,

The Square of the Number represented by d is dd, or	400
The Product of the two Numbers represented by d and m is dm, or 260, and twice that Product is $2\,dm$, or	520
Which 520 substracted from 400, leaves $dd - 2\,dm$, or $-$ 120, (see the Numerical Work in Question 6.)	-120
The Square of the Number represented by m is mm, or	169
Which 169 added to $-$120, makes $dd - 2\,dm + mm$, or $+$ 49, (see the Numerical Work in Question 6.)	49
To which adding the Number represented by b -	11
The Sum is 60, which I say is $= a$, or the Age of the Person - - - -	60

And is proved from the Conditions of the Question, thus:

I say the Person was - -	60 Years old
For if from that you substract - -	11
There remains - - -	49
The square Root of which is - -	7
To which adding - - -	13
The Sum is what the Question requires	20

To reduce an Equation by Evolution.

50. This is done by the Extraction of Roots, for if after all the known Quantities have been carried to the other Side of the Equation from the unknown Quantity, it appears that one Side of the Equation is the *Square*, *Cube*, or any Power of the unknown Quantity, then extract such Root of both Sides of the Equation as will depress or lower this Power of the unknown Quantity to the *first* Power; that is, if one Side of the Equation is the *Square* of the unknown Quantity, then the *Square Root* must be extracted; and if it is the *Cube* of the unknown Quantity, then the *Cube Root* must be extracted, and so on, which depressing the unknown Quantity to the first Power, the Question is answered.

Question 25. *What is that Number, if to the Square of which there is* 51 *added, the Sum may be* 100?

Let $a =$ the Number sought, $b = 51$, $m = 100$.

Now

Now there is a Number fought, which I call - - -	1	a
The Square of which by Art. 31. is	2	aa
To which 51, or b, being added, we have - - -	3	$aa+b$
And this $aa+b$ is by the Queftion to be equal to 100, or m; hence	4	$aa+b=m$
The Queftion being expreffed in *Algebra*, begin and tranfpofe b; then	5	$aa=m-b$
The known Quantities being now all on one Side of the Equation, and the other Side being aa, or the Square of a; therefore by the Rule extract the fquare Root of both Sides of the Equation. Now the fquare Root of aa is a, by Art. 33. and the Square Root of $m-b$ is $\sqrt{m-b}$, by Art. 34. and as the fquare Root of equal Quantities muft be equal, therefore	6	$a=\sqrt{m-b}$

Hence a, or the Number fought, is equal to the Number reprefented by m, fubftracting from it the Number reprefented by b, and extracting the fquare Root of the Remainder.

The Number reprefented by m, is - -	100
From which fubftracting b, or - - -	51
There remains $m-b$, or - - -	49
The fquare Root of which is $\sqrt{m-b}$, or 7, and is equal to a, or the Number fought - -	7

And is thus proved:

I fay the Number fought is - - -	7
The Square of which is - - -	49
To which adding - - - -	51
The Sum is what the Queftion requires -	100

Queftion 26. *A Merchant had gained fo many Pounds, that if from the Square of their Number is fubftracted* 101, *and to the Remainder add* 500, *this Sum will be* 3000 *Pounds. What had the Merchant gained?*

Let $a =$ the Gain of the Merchant, $b = 101$, $m = 500$, $p = 3000$.

Then a Merchant had gained a certain Number of Pounds, called -	1	a
The Square of which is by Art. 31.	2	aa
From which fubftracting 101, or b, we have - - -	3	$aa - b$
To which adding 500, or m, we have	4	$aa - b + m$
This, by the Queftion, is to be equal to 3000, or p; hence -	5	$aa - b + m = p$
By tranfpofing m we have -	6	$aa - b = p - m$
By tranfpofing b it is - -	7	$aa = p - m + b$
By extracting the fquare Roots, as at the fixth Step of the laft Example; then	8	$a = \sqrt{p - m + b}$

That is, a is equal to the Number reprefented by p, fubftracting from it the Number reprefented by m, and adding to this Remainder the Number reprefented by b, and extracting the fquare Root of the Sum.

The Number reprefented by p is - -	3000
From which fubftracting m, or - -	500
There remains $p - m$, or - - -	2500
To which adding b, or - - -	101
The Sum is $p - m + b$, or - -	2601
The fquare Root of which is $\sqrt{p - m + b}$, or 51, equal to a, the Number fought -	51

And is thus proved, from the Conditions of the Queftion.

I fay the Merchant gained - -	51 Pounds
For the Square of that is - -	2601
From which fubftracting - -	101
There remains - - -	2500
To which adding - -	500
The Sum is what the Queftion requires -	3000

Queftion 27. *If to the Square of the Number of Miles a Perfon had travelled there is added* 97, *fubftracting from the Sum* 251, *and adding to the Remainder* 160, *this Sum will be* 10006. *How many Miles had he travelled?*

Let

Let $a =$ the Number of Miles he had travelled, $b = 97$, $m = 251$, $x = 160$, $z = 10006$.

Then a Perſon had travelled a certain Number of Miles, called	1	a
The Square of which is by Article 31. - -	2	aa
To which adding 97, or b, it is	3	$aa + b$
From which ſubſtracting 251, or m, gives - -	4	$aa + b - m$
To which adding 160, or x, it is	5	$aa + b - m + x$
Which by the Queſtion is to be equal to 10006, or z, whence	6	$aa + b - m + x = z$
By tranſpoſing x it is -	7	$aa + b - m = z - x$
By tranſpoſing m we have -	8	$aa + b = z - x + m$
By tranſpoſing b then -	9	$aa = z - x + m - b$
By extracting the ſquare Root, as at the eighth Step of the laſt Example, we have -	10	$a = \sqrt{z - x + m - b}$

That is, from the Number repreſented by z, ſubſtract the Number repreſented by x, to the Remainder add the Number repreſented by m, from which Sum ſubſtract the Number repreſented by b, extract the ſquare Root of the Remainder, and it will be the Number ſought.

The Number repreſented by z is - -	10006
From which ſubſtracting x, or - -	160
There remains $z - x$, or - -	9846
To which adding m, or - -	251
The Sum is $z - x + m$, or - -	10097
From which ſubſtracting b, or - - -	97
There remains $z - x + m - b$, or -	10000
The ſquare Root of which, or $\sqrt{z - x + m - b}$, is the Number ſought - - -	100

PROOF.

PROOF.

I say the Person had travelled - -	100 Miles
For the Square of that is - -	10000
To which adding - -	97
The Sum is - - -	10097
From which substracting - -	251
There remains - - -	9846
To which adding - -	160
The Sum is what the Question requires -	10006

Question 28. *A General, upon numbering his Army, found, that if from the Square of the Number of Men in his Army, there was substracted* 3196, *and to the Remainder adding* 2721, *from which Sum substracting* 1711, *there would remain* 99997814. *To find the Number of Men in the Army?*

Let $a =$ the Number of Men in the Army, $b = 3196$, $m = 2721$, $x = 1711$, $z = 99997814$.

The Number of Men in the Army was - -	1	a
The Square of which is by Article 31. - -	2	aa
From which substracting 3196, or b, it is - -	3	$aa - b$
To which adding 2721, or m, gives - -	4	$aa - b + m$
From which substracting 1711, or x, we have - -	5	$aa - b + m - x$
Which by the Question is equal to 99997814, or z; hence	6	$aa - b + m - x = z$
First, by transposing x - -	7	$aa - b + m = z + x$
By transposing m - -	8	$aa - b = z + x - m$
By transposing b - -	9	$aa = z + x - m + b$
By extracting the square Root, as in the former Examples -	10	$a = \sqrt{z + x - m + b}$

By Numbers thus:

z is

z is in Numbers - - -	99997814
To which adding x, or - - -	1711
The Sum is $z+x$, or - -	99999525
From which ſubſtracting m, or - -	2721
There remains $z+x-m$, or - -	99996804
To which adding b, or - - -	3196
The Sum is $z+x-m+b$, or - -	100000000
The ſquare Root of which is a, or the Number ſought - - -	10000

Which is thus proved:

I ſay the Number of Men in the Army were	10000
For the Square of that is - -	100000000
From which ſubſtracting - -	3196
There remains - - -	99996804
To which adding - - -	2721
The Sum is - - -	99999525
From which ſubſtracting - -	1711
There remains what the Queſtion requires -	99997814

51. Theſe being the particular Methods by which Equations are reduced, or Queſtions anſwered, we ſhall now add ſome Examples where all theſe Methods are promiſcuouſly uſed.

Queſtion 29. *A Merchant broke for ſo many Pounds, that if their Number was multiplied by* 4, *and the Product divided by* 6, *and extracting the ſquare Root of the Quotient, from which ſubſtracting* 60, *there remains* 40. *What was the Sum for which the Merchant broke?*

Let $a=$ the Number of Pounds ſought, $b=4$, $d=6$, $m=60$, $p=40$.

Then the Merchant broke for a Number of Pounds, called	1	a
Which multiplied by 4, or b, we have - -	2	ba
This divided by 6, or d, we have - -	3	$\frac{ba}{d}$

The

The ſquare Root of which is by Art. 33. -	4	$\sqrt{\frac{ba}{d}}$
From this ſubſtracting 60, or m, we have -	5	$\sqrt{\frac{ba}{d}}-m$
Which by the Queſtion is equal to 40, or p, hence	6	$\sqrt{\frac{ba}{d}}-m=p$
Becauſe m is not under the radical Sign, therefore tranſpoſe it, by Art. 49.	7	$\sqrt{\frac{ba}{d}}=p+m$
Now ſquaring both Sides of the Equation by Art. 49.	8	$\frac{ba}{d}=pp+2pm+mm$
And multiplying by d, by Art. 47. then -	9	$\frac{dba}{d}=dpp+2dpm+dmm$
Rejecting d from $\frac{dba}{d}$, and putting down the other Quantities without any Alteration, as at Art. 47. we have -	10	$ba=dpp+2dpm+dmm$
Dividing by b, by Art. 48. then - -	11	$\frac{ba}{b}=\frac{dpp+2dpm+dmm}{b}$
Rejecting b from $\frac{ba}{b}$, and putting down the other Quantities without any Alteration, as at Art. 47, or 48, we have	12	$a=\frac{dpp+2dpm+dmm}{b}$

In Numbers thus:

$$\begin{aligned} dpp &= 9600 \\ +2dpm &= 28800 \\ +dmm &= 21600 \\ \text{Sum } & \overline{60000} \text{ or } dpp+2dpm+dmm \end{aligned}$$

Now dividing 60000, or $dpp+2dpm+dmm$, by 4, or b, we have $\frac{dpp+2dpm+dmm}{b}$, or 60000, divided by $4=15000$, which is equal to a, or the Number of Pounds for which the Merchant broke.

PROOF.

PROOF.

$$\begin{array}{r}15000\\ 4\\ \hline 6)60000\\ \hline 10000\end{array}$$

(100 the ſquare Root of 10000
60
40 as the Queſtion requires.

Queſtion 30. *A Gentleman having bought a Houſe, and being diſpoſed to try the Knowledge of his Son in* Algebra, *told him, if the Number of Pounds the Houſe coſt was divided by* 8, *and that Quotient multiplied by* 50, *and extracting the ſquare Root of the Product, to which adding* 10, *this Sum would be* 60 *Pounds. What did the Houſe coſt?*

Let $a =$ the Price of the Houſe, $b = 8$, $d = 50$, $m = 10$, $p = 60$.

Now the Price of the Houſe is -	1	a
Which divided by 8, or b, it is - -	2	$\frac{a}{b}$
This multiplied by 50, or d, we have -	3	$\frac{da}{b}$
The ſquare Root of which is, by Art. 33. -	4	$\sqrt{\frac{da}{b}}$
To which adding 10, or m	5	$\sqrt{\frac{da}{b}}+m$
This, by the Queſtion, is equal to 60, or p, hence	6	$\sqrt{\frac{da}{b}}+m=p$
The Queſtion being now expreſſed in *Algebra*, and m not being under the radical Sign, tranſpoſe it by Art. 49. then -	7	$\sqrt{\frac{da}{b}}=p-m$
Now ſquaring both Sides of the Equation, by Art. 49.	8	$\frac{da}{b}=pp-2pm+mm$
And multiplying by b, by Art. 47. - -	9	$\frac{bda}{b}=bpp-2bpm+bmm$

Rejecting b from $\frac{b\,d\,a}{b}$, and putting down the rest as at the twelfth Step of the last Question, then	10	$d\,a = b\,p\,p - 2\,b\,p\,m + b\,m\,m$
Dividing by d, by Art. 48. then - -	11	$\frac{d\,a}{d} = \frac{b\,p\,p - 2\,b\,p\,m + b\,m\,m}{d}$
Rejecting d from $\frac{d\,a}{d}$, and putting down the rest as at Art. 47, or 48. and	12	$a = \frac{b\,p\,p - 2\,b\,p\,m + b\,m\,m}{d}$

In Numbers:

$$
\begin{array}{rr}
b\,p\,p = & 28800 \\
-2\,b\,p\,m = - & 9600 \\
\hline
 & 19200 \\
+b\,m\,m = & 800 \\
\hline
d = 5|0) & 2000|0 \\
\hline
 & 400 = a,
\end{array}
$$

the Number of Pounds the House cost.

PROOF.

$$
\begin{array}{r}
8)400 \\
\hline
50 \\
50 \\
\hline
2500 \\
10 \\
\hline
60
\end{array}
$$

(50 the Square Root of 2500

60 as the Question requires.

CONSECTARY.

If the Reader compares the eighth, ninth, and tenth Steps of the last Work, he will find that to multiply any Fraction by its Denominator, or any Dividend by its Divisor, is only to reject the Denominator, or Divisor, from that Quantity, and multiply it into all the other Quantities; thus, the Equation at the eighth Step is $\frac{d\,a}{b} = p\,p - 2\,p\,m + m\,m$, which being multiplied by its

its Denominator b, we have at the tenth Step $da = bpp - 2bpm + bmm$; the ninth Step, or $\frac{bda}{b} = bpp - 2bpm + bmm$, being only a more particular Illuſtration of the Work.

And by comparing the tenth, eleventh, and twelfth Steps of the ſame Work, it appears, that to divide any Quantity, by any Letter in that Quantity, is only to reject that Letter from the Quantity, and placing it as a Diviſor to the other Quantities; thus, at the tenth Step, the Equation is $da = bpp - 2bpm + bmm$, which being divided by d, gives at the twelfth Step $a = \frac{bpp - 2bpm + bmm}{d}$; the eleventh Step or $\frac{da}{d} = \frac{bpp - 2bpm + bmm}{d}$, being only a more particular Illuſtration of the Work.

Therefore we ſhall for the future leave out ſuch Steps as the ninth and eleventh: I did not chooſe to do it before, my Deſign being to make this curious Science as eaſy as poſſible.

Queſtion 31. *A Running-Footman being ſent of an Errand, was told, that if he ſquared the Number of Miles he was to run, and multiplied the Square by* 4, *and divided the Product by* 40, *to this Quotient adding* 500, *from which Sum ſubſtracting* 1400, *and extracting the ſquare Root of the Remainder, it would be* 10. *How many Miles was the Footman to run?*

Let $a =$ the Number of Miles the Footman was to run, $b = 4$, $d = 40$, $m = 500$, $x = 1400$, $p = 10$.

The Number of Miles the Footman was to run let be -	1	a
Which being ſquared is by Art. 31. - -	2	aa
This being multiplied by 4, or b, we have -	3	baa
This being divided by 40, or d, it is - -	4	$\frac{baa}{d}$
To which adding 500, or m, gives - -	5	$\frac{baa}{d} + m$

From which ſubſtracting 1400, or x, we have -	6	$\frac{baa}{d}+m-x$
The ſquare Root of which is by Art. 34. -	7	$\sqrt{\frac{baa}{d}+m-x}$
Which by the Queſtion, is equal to 10, or p, therefore -	8	$\sqrt{\frac{baa}{d}+m-x}=p$
Now ſquare both Sides of the Equation, by Art. 49. and	9	$\frac{baa}{d}+m-x=pp$
By tranſpoſing x, we have -	10	$\frac{baa}{d}+m=pp+x$
By tranſpoſing m, we have	11	$\frac{baa}{d}=pp+x-m$
By multiplying by d by the *Conſectary*, Page 130. -	12	$baa=dpp+dx-dm$
And dividing by b by the *Conſectary*, Page 130. -	13	$aa=\frac{dpp+dx-dm}{b}$
Now extracting the ſquare Root, by Art. 50. - -	14	$a=\sqrt{\frac{dpp+dx-dm}{b}}$

In Numbers:

$$\begin{array}{rr} dpp = & 4000 \\ +dx = & 56000 \\ \hline & 60000 \\ -dm = & -20000 \\ \hline b = & 4)\ 40000 \\ \hline & 10000 \end{array}$$

(100 the ſquare Root of 10000, hence the Footman was to run 100 Miles.

PROOF.

$$\sqrt{\frac{4aa}{40}+500-1400}=10$$

I have not drawn out the Proof of the laſt Queſtion into Particulars, but only expreſſed it at once; that is, four times the Square of a (which is found to be 100) being divided by 40, if to this Quotient we add 500, and from this Sum ſubſtract 1400, the ſquare Root of this Remainder will be equal to 10. And now I ſhall expreſs all the Conditions of the Queſtion at the firſt, Equation,

Equation, that the Learner may form ſome little Judgment in what Manner to ſhorten his Work; and if he conceives how the Proof of the laſt Queſtion is expreſſed, it will eaſily lead him to the Knowledge of expreſſing the Conditions of the Queſtion, or raiſe ſuch Equations as ariſe from the Queſtion without particularizing every Circumſtance. But if the Learner finds any Difficulty in this, he may proceed as before.

Queſtion 32. *A Gentleman who had been at the Gaming-Tables, and loſing, ſome of his Acquaintance laughing at him for his Folly, aſked how much he had loſt; to which he anſwered, if you ſquare the Number of Pounds I have loſt, and divide that by* 4, *multiplying this Quotient by* 10, *to which Product add* 3900, *then extracting the ſquare Root of this Sum, from which ſubſtracting* 80, *the Remainder will be equal to* 90. *How much had he loſt?*

Let a = the Number of Pounds loſt, $b = 4$, $d = 10$, $m = 3900$, $p = 80$, $z = 90$.

Then by the Queſtion	1	$\sqrt{\frac{daa}{b}+m}:-p=z$
By tranſpoſing p, it not being under the radical Sign, by Art. 49. we have -	2	$\sqrt{\frac{daa}{b}+m}=z+p$
By ſquaring both Sides of the Equation, by Art. 49. then -	3	$\frac{daa}{b}+m=zz+2zp+pp$
By tranſpoſing m, it is	4	$\frac{daa}{b}=zz+2zp+pp-m$
Multiplying by b by the Confectary, Page 130. -	5	$daa=bzz+2bzp+bpp-bm$
Dividing by d by the ſame - -	6	$aa=\frac{bzz+2bzp+bpp-bm}{d}$
Extracting the ſquare Root, by Art. 50.	7	$a=\sqrt{\frac{bzz+2bzp+bpp-bm}{d}}$

In Numbers:

$$\begin{array}{rr} bzz = & 32400 \\ 2bzp = & 57600 \\ bpp = & 25600 \\ \hline & 115600 \\ -bm = - & 15600 \\ \hline d = 1|0) & 10000|0 \\ \hline & \dot{1}0\dot{0}0\dot{0}(100 = a, \end{array}$$

the Number of Pounds loſt.

$$\begin{array}{r} 1 \\ \hline 0 \end{array}$$

PROOF.

$$\sqrt{\frac{10aa}{4} + 3900} : - 80 = 90$$

To reduce an Equation when the unknown Quantity is in ſeveral Terms.

52. When the unknown Quantity is in more Terms than one, bring all thoſe Terms which have the unknown Quantity to one Side of the Equation, taking Care that the *greateſt Co-efficient* of the unknown Quantity has at laſt the *affirmative* Sign, and carrying all the Quantities that are known on the other Side of the Equation; then divide both Sides of the Equation by *all the Co-efficients* of the unknown Quantity, connected with the ſame Signs of + and —, as they then happen to have, which will reduce the Equation, as in the following Examples.

If the unknown Quantity ſhould be in more than two Terms, tranſpoſe thoſe Terms in ſuch a Manner, that the Sum of the *poſitive* Co-efficients of the unknown Quantity may exceed the Sum of the *negative* Co-efficients of the unknown Quantity, and then divide as before directed.

Queſtion 33. *There is a certain Number which being multiplied by* 10, *if this Product is divided by* 2, *to this Quotient adding* 19, *and ſubſtracting* 99 *from that Sum, the Remainder will be equal to the Number ſought.*

Let $a =$ the Number ſought, $b = 10$, $d = 2$, $m = 19$, $z = 99$.

By the Queſtion - -	1	$\frac{ba}{d}+m-z=a$
By tranſpoſing z - -	2	$\frac{ba}{d}+m=a+z$
By tranſpoſing m -	3	$\frac{ba}{d}=a+z-m$
By multiplying by d by the *Conſectary*, Page 130. -	4	$ba=da+dz-dm$
Becauſe d is leſs than b, tranſpoſe da, that both the Terms which have the unknown Quantity, may be on the ſame Side of the Equation, then	5	$ba-da=dz-dm$
And dividing according to the Rule by $b-d$, the two Coefficients of a, we have - -	6	$a=\frac{dz-dm}{b-d}=20$ the Number ſought.

PROOF.

$$\frac{10a}{2}+19-99=a$$

The Diviſion at the fifth and ſixth Steps, *viz.* that $ba-da$, divided by $b-d$, ſhould leave only a, may perhaps a little perplex the Learner; and if it does, I adviſe him to examine *Art.* 10. where he may obſerve, that in multiplying any compound Quantity by a ſingle Letter, that Letter goes into every Term of the Product, therefore the Multiplier is not ſo many Times that Letter as the Number of Terms are in which that Letter is found, but only that ſingle Letter multiplied ſucceſſively into all the other Quantities; hence, if this Product is to be divided by all thoſe Quantities, the Quotient will be the ſingle Letter, and not ſo many Times that Letter as the Number of Terms are in which it is found. See farther the Proof of the Queſtion 38. and Art. 22.

Queſtion 34. *A Gentleman bought an Eſtate for ſo many Pounds, that if they were multiplied by* 4, *and this Product divided by* 5, *from which Quotient ſubſtracting* 600, *and adding to the Remainder* 6 *Times what the Eſtate coſt, this Sum will be equal to* 6200 *Pounds. How much did the Eſtate coſt?*

Let

Let $a =$ the Number of Pounds the Eſtate coſt, $b = 4$, $d = 5$, $m = 600$, $p = 6$, $x = 6200$.

Then by the Queſtion -	1	$\frac{ba}{d} - m + pa = x$
By tranſpoſing m, we have	2	$\frac{ba}{d} + pa = x + m$
Multiplying by d by the *Conſectary*, Page 130 -	3	$ba + dpa = dx + dm$
Dividing by $b + dp$, the Co-efficients of a, as in the laſt Queſtion, and we have -	4	$a = \frac{dx + dm}{b + dp} = 1000$

Therefore the Eſtate coſt 1000 Pounds.

PROOF.

$$\frac{4a}{5} - 600 + 6a = 6200$$

Queſtion 35. *A Perſon had a certain Number of Shillings, which multiplied by* 4, *this Product being divided by* 11, *to the Quotient adding* 90, *and from this Sum taking away* 30, *the ſquare Root of this Remainder will be equal to the ſquare Root of the Number of Shillings ſought, after being diminiſhed by* 10.

Let $a =$ the Number of Shillings ſought, $b = 4$, $d = 11$, $x = 90$, $p = 30$, $z = 10$.

Then by the Queſtion -	1	$\sqrt{\frac{ba}{d} + x - p} = \sqrt{a - z}$
Becauſe there is no Quantity on each Side of the Equation but what is under the radical Sign, therefore ſquare both Sides of the Equation, by Art. 49.	2	$\frac{ba}{d} + x - p = a - z$
Multiplying by d by the *Conſectary*, Page 130. -	3	$ba + dx - dp = da - dz$
Becauſe d, one Co-efficient of a, is greater than b, the other Co-efficient of a, tranſpoſe $b\,a$, then - -	4	$dx - dp = da - dz - ba$

Tranſpoſing

Tranpofing dz - -	5	$dz + dx - dp = da - ba$
Dividing by $d - b$, the two Co-efficients of a, as at Queftion 33. Step 6. we have - - -	6	$a = \frac{dz + dx - dp}{d - b} = 110$ (the Number fought.

If the Learner chooses to have the unknown Quantity on the left Side of the Equation, he might have put the 5th Step thus, $da - ba = dz + dx - dp$, this being only to change the Sides of the Equation, not to alter their Value.

PROOF.

If $a = 110$, then $\sqrt{\frac{4a}{11} + 90 - 30} = \sqrt{a - 10}$

Queftion 36. *A Running-Footman forward to fhow his Learning, being in Company, faid, if the Number of Miles he had run was multiplied by* 7, *to which Product adding* 550, *and fubftracting* 20 *from that Sum, and dividing the Remainder by* 10, *the fquare Root of the Quotient will be the fame, as if you added* 14 *Miles to thofe he had run, and extracted the fquare Root of that Sum.*

Let $a =$ the Number of Miles he had run, $b = 7$, $d = 550$, $m = 20$, $p = 10$, $x = 14$.

Then by the Queftion -	1	$\sqrt{\frac{ba + d - m}{p}} = \sqrt{a + x}$
There being no Quantity without the radical Sign, therefore fquare both Sides of the Equation as at the fecond Step of the laft Queftion -	2	$\frac{ba + d - m}{p} = a + x$
Multiplying by p by the *Confectary*, Page 130.	3	$ba + d - m = pa + px$
Becaufe p, one Co-efficient of a, is greater than b, the other Co-efficient of a, therefore tranfpofe ba	4	$d - m = pa + px - ba$
By tranfpofing px -	5	$d - m - px = pa - ba$
Dividing by $p - b$, the two Co-efficients of a, as at Queftion 33. Step. 6. we have -	6	$a = \frac{d - m - px}{p - b} = 130$ the (Number of Miles required.

PROOF.

PROOF.

$$\sqrt{\frac{7a + 550 - 20}{10}} = \sqrt{a + 14}$$

To reduce an Equation when the ſame Quantity, either known or unknown, is in every Term of the Equation.

53. In any Algebraic Operation, if the ſame Quantity, either known or unknown, is in every Term of any Equation, *then divide every Term of the Equation by that Quantity, which will reduce the Equation to more ſimple Terms,* as in the following Queſtions.

Queſtion 37. *To find a Number which multiplied by 4, and the Product added to the ſame Number multiplied by 56, and divided by 7, this Sum will be equal to the Square of the Number ſought.*

Let $a =$ the Number ſought, $b = 4$, $d = 56$, $m = 7$.

Then by the Queſtion -	1	$ba + \frac{da}{m} = aa$
Multiplying by m by the *Conſectary*, Page 130.	2	$mba + da = maa$
Becauſe a is in every Term of the Equation, divide by a, then - -	3	$mb + d = ma$
Dividing by m, the Coefficient of a, by the *Conſectary*, Page 130. and	4	$a = \frac{mb + d}{m} = 12$ the Number ſought.

PROOF.

$$4a + \frac{56a}{7} = aa$$

Queſtion 38. *There are two Towns at ſuch a Diſtance, that if the Number of Miles between them is multiplied by 79, and this Product added to their Diſtance, the ſquare Root of this Sum will be equal to the Diſtance of the two Towns multiplied by 2.*

Let

Let $a =$ the Diſtance of the Towns, $b = 79$, $m = 2$.

Then by the Queſtion - -	1	$\sqrt{ba+a} = ma$
There being no rational Quantities on the ſame Side of the Equation where the radical Sign is, ſquare both Sides of the Equation, and - -	2	$ba+a = mmaa$
Dividing by a, it being in every Term of the Equation, and - -	3	$b+1 = mma$
Dividing by mm, the Co-efficient of a	4	$a = \frac{b+1}{mm} = 20$

Hence the Diſtance between the two Towns is 20 Miles.

PROOF.

$$\sqrt{79a+a} = 2a$$

If the Reader does not eaſily conceive that dividing $ba+a$, or $ba+1a$ at the ſecond Step, by a, gives $b+1$, as at the third Step, I adviſe him to conſider what is ſaid at Queſtion 33; to which may be added, that $\overline{b+1} \times a = ba+a$, whereas $\overline{b+1} \times 2a = 2ba+2a$, a Product very different from $ba+a$. Or it may be explained thus, $\frac{ba+1a}{a} = b+1$, the a being rejected by Art. 22 and 26.

The Manner of regiſtering the Steps of an Algebraic *Operation explained.*

54. Having explained to the young *Analyſt* the different Methods of managing Equations, to ſave the Trouble of uſing ſo many Words, I ſhall now ſhow him the Method of *regiſtering the Steps*, introduced by the ingenious Dr. *John Pell.*

To *regiſter the Steps* of an Analytic Operation is only to expreſs in the Margent of the Work by *Symbols*, inſtead of Words, what has been done; and to render it as eaſy as may be to the Learner, we ſhall reſume the Work of one of the former Queſtions, and expreſs by Words what is done in one Column, in another Column expreſs the ſame Thing by *Symbols*, or Characters, and in the third Column place the Work itſelf, that by comparing the Operation with the Obſervations that follow it, the Reader may the more eaſily underſtand the Manner of *regiſtering the Steps.*

Queſtion 39. *A Running-Footman being ſent of an Errand, was told, that if he ſquared the Number of Miles he was to run, and multiplied it by* 4, *and divided the Product by* 40, *to this Quotient adding* 500, *from which Sum ſubſtracting* 1400, *and extracting the ſquare Root of the Remainder, it would be* 10. *How many Miles was the Footman to run?* (this is Queſt. 31.)

Let $a =$ the Number of Miles the Footman was to run, $b = 4$, $d = 40$, $m = 500$, $x = 1400$, $p = 10$.

	Regiſter		
Then, by the Queſtion, we have the ſame Equation as at the eighth Step, Queſtion 31.		1	$\sqrt{\frac{baa}{d} + m - x} = p$
Squaring both Sides of the Equation, or involving them to the ſecond Power, by Art. 49. -	$1 \odot 2$	2	$\frac{baa}{d} + m - x = pp$
By tranſpoſing x at the ſecond Equation	$2 + x$	3	$\frac{baa}{d} + m = pp + x$
By tranſpoſing m at the third Equation -	$3 - m$	4	$\frac{baa}{d} = pp + x - m$
Multiplying the fourth Equation by d -	$4 \times d$	5	$baa = dpp + dx - dm$
Dividing the fifth Equation by b - -	$5 \div b$	6	$aa = \frac{dpp + dx - dm}{b}$
Extracting the ſquare Root of the ſixth Equation, by Art. 50.	6 ɯ 2	7	$a = \sqrt{\frac{app + dx - dm}{b}} = 10$

For another Inſtance let us take Queſtion 33.

Queſtion 40. *There is a certain Number, which being multiplied by* 10, *if this Product is divided by* 2, *to this Quotient adding* 19, *and ſubſtracting* 99 *from that Sum, the Remainder will be equal to the Number ſought.*

Let $a =$ the Number ſought, $b = 10$, $d = 2$, $m = 19$, $z = 99$.

Then

	Register		
Then by the Queſtion - -		1	$\frac{ba}{d}+m-z=a$
By tranſpoſing z from the firſt Equation -	$1+z$	2	$\frac{ba}{d}+m=a+z$
By tranſpoſing m from the ſecond Equation -	$2-m$	3	$\frac{ba}{d}=a+z-m$
Multiplying the third Equation by d -	$3\times d$	4	$ba=da+dz-dm$
By tranſpoſing da from the fourth Equation	$4-da$	5	$ba-da=dz-dm$
Dividing the fifth Equation by $b-d$, the two Co-efficients of a, by Art. 52.	$5\div b-d$	6	$a=\frac{dz-dm}{b-d}=20$

From theſe two Examples we may obſerve, that to *regiſter* any Operation, is only to put down the Figure which ſtands in the Column againſt that Equation, from which we intend to raiſe the next Equation, and after that the *Sign* of either *Addition*, *Subſtraction*, *Multiplication*, *Diviſion*, *Involution* and *Evolution*, according as the Caſe requires, and after this the Quantity which ſuffers the Alteration.

Thus at Queſtion 39, the firſt Equation being raiſed or involved to the ſecond Power produces the ſecond Equation, therefore, I ſay in the *Regiſter* 1 ☉ 2, that is, the firſt Equation involved to the ſecond Power gives the ſecond Equation, and in the ſame Operation.

Becauſe the fourth Equation is produced from the third, by tranſpoſing m with the Sign $-$, therefore in the *Regiſter* I ſay $3-m$, that is, the third Equation $-m$, produces the fourth Equation. And,

As the fifth Equation is produced from the fourth by multiplying by d, therefore I ſay in the *Regiſter* $4\times d$, that is, the fourth Equation multiplied by d, produces the fifth Equation. And,

As the ſixth Equation is produced from the fifth by dividing by b, therefore, I ſay in the *Regiſter* $5\div b$, that is, the fifth Equation divided by b, produces the ſixth Equation. And,

As the ſeventh Equation is produced from the ſixth by extracting the ſquare Root, I ſay in the *Regiſter* 6 ɯ 2, that is, the ſixth Equation having the Square Root extracted, produces the ſeventh Equation.

Whence, as I ſaid above, to regiſter any Operation, is only to put down whether it is the firſt, ſecond, third, fourth, or any other Equation, which ſuffers the Alteration, and from which the new Equation is raiſed; and after that Figure to expreſs in Characters, or Signs, the Alteration that is then made to gain the new Equation.

The Method of reſolving Queſtions that contain two Equations, and two unknown Quantities.

55. THE foregoing Queſtions requiring only one unknown Number to be found, their Conditions were all expreſſed in one Equation, which Equation being reduced by the Rules already delivered, the Queſtion was anſwered.

But if the Queſtion requires two unknown Quantities to be found, then there are generally raiſed two Equations from the Queſtion, each of them including both the unknown Quantities; which may be reſolved by this

RULE.

Find what the ſame unknown Quantity is *equal* to in each of the two Equations, which ariſe from the Conditions of the Queſtion, then make theſe two Equations *equal* to one another, and in this Equation there will be but *one unknown Quantity*, conſequently if this Equation is reduced by the Rules already given at Art. 46 to 53. we ſhall find what this unknown Quantity is.

And to find the Value of an unknown Quantity in any Equation, is only to find what it is equal to, therefore all the other Quantities, whether known or unknown, muſt be carried to the other Side of the Equation by the Directions at Art. 46 to 53. and then it will appear to what this unknown Quantity is equal, as this makes one Side of the Equation, the other Side of the Equation being known Quantities, with the other unknown Number or Quantity ſought.

Finding

Finding the Value of the same unknown Quantity in each of the given Equations, and making these two Equations equal to one another, which clears the Work of that unknown Quantity whose Value was found, is called *exterminating an unknown Quantity.*

Question 41. *To find two Numbers, if the greater is added to the lesser, the Sum is* 262.

But if from the greater you substract the lesser, the Remainder is 144.

Let $a =$ the greater Number, and $e =$ the lesser Number sought, $b = 262$, $x = 144$.

Now the Sum of the two Numbers in *Algebra* is $a + e$, which is equal to 262, or b, hence we have - - -	1	$a + e = b$	By the Question
And the lesser Number being substracted from the greater is $a - e$, which is equal to 144, or x, hence we have -	2	$a - e = x$	

The Conditions of the Question being now expressed, there appears in the above two Equations, two unknown Quantities a and e, therefore according to the Rule find what a is equal to in the first Equation, by transposing e.

$1 - e$	3	$a = b - e$
Now find what a is equal to in the second Equation, by transposing e - -		
$2 + e$	4	$a = x + e$

Therefore make the third and fourth Equations equal to one another, for they are both equal to the same Quantity a, which exterminates that unknown Quantity: this Step is *registered* by placing the 3 and 4 with a Point between them as in the Work, which expresses that the fifth Equation is from comparing the third and fourth Equation together

$3 \cdot 4$	5	$x + e = b - e$

The

The unknown Quantity e being on both Sides of the Equation, bring it on one Side of the Equation, by Art. 52.

$5 + e$	6	$x + 2e = b$
$6 - x$	7	$2e = b - x$
$7 \div \bar{2}$	8	$e = \dfrac{b - x}{2}$

Here it appears that e, or the leſſer Number ſought, is equal to b, or 262, ſubſtracting from it x, or 144, and dividing the Remainder by 2.

When any Equation is divided by an abſolute Number, as the ſeventh Equation is divided by 2, place them in the *Regiſter* as uſual, but draw a Line over the 2 to diſtinguiſh that it is an abſolute Number by which you divide, and not by the ſecond Equation in the Work.

$$\begin{array}{rl} \text{Now } b = & 262 \\ -x = & -144 \\ \hline & 2)\,118 \\ \hline & 59 = e, \end{array}$$ the leſſer of the two Numbers ſought.

It being now known what e is in Numbers, we may find a by the third or fourth Equation, and by the third Equation we have $a = b - e$.

$$\begin{array}{rl} \text{But } b = & 262 \\ -e = & -59 \\ \hline & 203 = a, \end{array}$$ the greater of the two Numbers ſought.

Whence 203 and 59 are the two Numbers required in the Queſtion, and is thus proved from their Conditions.

The greater Number is	203	- -	203
The leſſer Number is	59	- -	59
	262 Sum		144 Remains.

Queſtion 42. *Two Men diſcourſing of their Money, found that if the Number of Shillings each had were added together, the Sum would be 38.*

But if from him that had the greater Number of Shillings, there be ſubſtracted twice the Number of Shillings the other Perſon had, there would remain 5. How many had each Man?

Let

Let $a =$ the greater Number of Shillings, $e =$ the lesser Number of Shillings, $b = 38$, $x = 5$.

And because the Sum of their Shillings, or $a + e$, was 38, or b, hence		1	$a + e = b$	By the Question
And twice the lesser Number being taken from the greater, or $a - 2e$, was equal to 5, or x, hence		2	$a - 2e = x$	By the Question
Now to find the Value of a in the first Equation, transpose e.	$1 - e$	3	$a = b - e$	
And to find the Value of a in the second Equation, transpose $2e$.	$2 + 2e$	4	$a = x + 2e$	

Make the third and fourth Equations equal to one another, because they are both equal to the same Quantity a, and *register* it as directed in the last Question; and this exterminates the unknown Quantity a.

$3 \cdot 4$	5	$x + 2e = b - e$

The unknown Quantity e being on both Sides of the Equation, bring it on one Side of the Equation, by Art. 52.

$5 + e$	6	$x + 3e = b$
$6 - x$	7	$3e = b - x$
$7 \div 3$	8	$e = \dfrac{b - x}{3}$

Hence the Question is answered, for $b = 38$

$$\begin{array}{r} -x = -\ 5 \\ \hline 3)\ 33 \\ \hline 11 = e, \end{array}$$

the lesser Number of Shillings.

And as e is now known, we may find what a is by the third or fourth Equation; taking the fourth Equation, we have

$$\begin{array}{r} x = 5 \\ 2e = 22 \\ \hline 27 = a, \end{array}$$ the greater Number of Shillings.

PROOF.

The greater Number of Shillings	27	- - -	27
The leſſer Number of Shillings	11	Twice the leſſer Number of Shillings	22
	Sum 38	Remains	5

Queſtion 43. *Two Men laying a Wager concerning the Number of Sheep in two Droves, as they could not decide it, appealed to a third Perſon, who told them, that if* 31 *was added to the Number of Sheep in the greateſt Drove, that Sum would be equal to twice the Number of Sheep in the leaſt Drove.*

But if they added 44 *to the Number of Sheep in the leaſt Drove, that Sum would be as many as were in the greateſt Drove, and deſired they would now find the Number of Sheep in each Drove.*

Let $a =$ the Number of Sheep in the greateſt Drove, $e =$ the Number of Sheep in the leaſt Drove, $x = 31$, $d = 44$.

Now the Number of Sheep in the greateſt Drove being added to 31, is equal to twice the Number of Sheep in the leſſer Drove, hence - -	1	$a + x = 2e$	By the Queſtion
And the Number of Sheep in the leaſt Drove when added to 44, being equal to the Number of Sheep in the greateſt Drove, we have - -	2	$e + d = a$	
$1 - x$	3	$a = 2e - x$	

Now by the third Equation, a is equal to $2e - x$, and by the ſecond Equation, a is equal to $e + d$, therefore make theſe Equations equal to one another, for they are both equal to the ſame Quantity a, which exterminates a, as before.

2 . 3

$2 \cdot 3$	4	$2e - x = e + d$
$4 - e$	5	$e - x = d$
$5 + x$	6	$e = d + x$

$d = 44$
$x = 31$
$75 = e$, the Number of Sheep in the leaſt Drove.

Then having found e, we may find a by the ſecond Equation.

$e = 75$
$d = 44$
$119 = a$, the Number of Sheep in the greateſt Drove.

PROOF.

119
31
150 which is twice the Number of Sheep in the leaſt Drove.

75
44
119 which is the Number of Sheep in the greateſt Drove.

Queſtion 44. *Two Gentlemen who had ſold their Eſtates, by comparing what each Eſtate was ſold for, found, that twice the Sum of what both the Eſtates was ſold for was* 11468 *Pounds:*

And if what the leaſt Eſtate was ſold for, be ſubſtracted from what the greateſt Eſtate was ſold for, there will remain 1408 *Pounds. For how much was each Eſtate ſold?*

Let $a =$ the Number of Pounds the greateſt Eſtate was ſold for, $e =$ the Number of Pounds the leaſt Eſtate was ſold for, $b = 11468$, $x = 1408$.

By the firſt Condition -	1	$2a + 2e = b$
By the ſecond Condition -	2	$a - e = x$
Find the Value of a, in the firſt Equation.		
$1 - 2e$	3	$2a = b - 2e$
$3 \div 2$	4	$a = \dfrac{b - 2e}{2}$

Now

Now find the Value of a, in the second Equation.

$2 + e \mid 5 \mid a = x + e$

Make the fourth and fifth Equations equal to one another, because they are both equal to the same Quantity a, and therefore must be equal to one another, by which a will be exterminated.

$4 \cdot 5$	6	$x + e = \frac{b - 2e}{2}$*
6×2	7	$2x + 2e = b - 2e$
$7 + 2e$	8	$2x + 4e = b$
$8 - 2x$	9	$4e = b - 2x$
$9 \div 4$	10	$e = \frac{b - 2x}{4} = 2163$, the Pounds for

which the least Estate was sold; and e being now known, then

* Here we have only to find what e is by the Rules already delivered, at Art. 46 to 53.

By the fifth Step | 11 | $a = x + e = 3571$, the Pounds for which the greatest Estate was sold.

PROOF.

Now if $a = 3571$, and $e = 2163$, then $2a + 2e = 11468$, and $a - e = 1408$.

Question 45. *Two Gamesters,* A *and* B, *found, that if twice the Number of Pounds won by* A *was added to what had been won by* B, *the Sum was* 48 *Pounds:*

And if what had been won by A *was added to three times what had been own by* B, *the Sum was* 39 *Pounds. What was the Sum won by each Gamester?*

Let $a =$ the Pounds won by A, $e =$ the Pounds won by B, $b = 48$, $x = 39$.

By the first Condition -	1	$2a + e = b$
By the second Condition -	2	$a + 3e = x$

Find

Find the Value of a, from the first Equation.

$1 - e$	3	$2a = b - e$
$3 \div 2$	4	$a = \frac{b - e}{2}$

Now find the Value of a, from the second Equation.

$2 - 3e$	5	$a = x - 3e$

Make the fourth and fifth Equations equal to one another, because they are each equal to the same Quantity, which Equation will exterminate a.

$4 \cdot 5$	6	$\frac{b - e}{2} = x - 3e$*
6×2	7	$b - e = 2x - 6e$
$7 + 6e$	8	$b + 5e = 2x$
$8 - b$	9	$5e = 2x - b$
$9 \div 5$	10	$e = \frac{2x - b}{5} = 6$ Pounds, won by B.
Then from the fifth Equation	41	$a = x - 3e = 21$ Pounds, won by A.

* Here we have only to find e by the Rules already delivered, at Art. 46 to 53.

PROOF.

$$2a + e = 48$$
$$a + 3e = 39$$

Question 46. *What are those two Numbers, that twice the greater being added to three times the lesser, the Sum is* 29:
And three times the greater being substracted from five times the lesser, the Remainder is 4.

Let $a =$ the greater Number, $e =$ the lesser Number, $b = 29$, $m = 4$.

By the first Condition	1	$2a + 3e = b$
By the second Condition	2	$5e - 3a = m$

Find

Find the Value of a, from the firſt Equation.

$1 - 3e$	3	$2a = b - 3e$
$3 \div 2$	4	$a = \dfrac{b - 3e}{2}$

Now find the Value of a, from the ſecond Equation; tranſpoſe $3a$, becauſe it has the *negative* Sign.

$2 + 3a$	5	$5e = m + 3a$
$5 - m$	6	$5e - m = 3a$
Or	7	$3a = 5e - m$
$7 \div 3$	8	$a = \dfrac{5e - m}{3}$

Make the fourth and eighth Equations equal to one another, for they are each equal to the ſame Quantity a, and this unknown Quantity will be exterminated.

$4 \cdot 8$	9	$\dfrac{5e - m}{3} = \dfrac{b - 3e}{2}$
9×2	10	$\dfrac{10e - 2m}{3} = b - 3e$
10×3	11	$10e - 2m = 3b - 9e$
$11 + 9e$	12	$19e - 2m = 3b$
$12 + 2m$	13	$19e = 3b + 2m$
$13 \div 19$	14	$e = \dfrac{3b + 2m}{19} = 5$, the leſſer Number.
By the fourth Equation	15	$a = \dfrac{b - 3e}{2} = 7$, the greater Number.

PROOF.

$$2a + 3e = 29$$
$$5e - 3a = 4$$

Queſtion 47. *Two Travellers,* A *and* B, *meeting on the Road, found, that if the Number of Miles travelled by* A *was divided by five, adding to this Quotient three times the Number of Miles travelled by* B, *the Sum was* 249:

But if twice the Number of Miles travelled by A *were added to four times the Number of Miles travelled by* B, *the Sum was* 540. *How many Miles had each travelled?*

Let a = the Number of Miles travelled by A, e = the Number of Miles travelled by B, $x = 249$, $z = 540$.

By the first Condition -	1	$\frac{a}{5} + 3e = x$
By the second Condition -	2	$2a + 4e = z$
1×5	3	$a + 15e = 5x$
$3 - 15e$	4	$a = 5x - 15e$

The Value of a being now found by the first Equation, find its Value from the second Equation.

$2 - 4e$	5	$2a = z - 4e$
$5 \div 2$	6	$a = \frac{z - 4e}{2}$

Now make the fourth and sixth Equations equal to one another as before, which exterminates a.

$4 \cdot 6$	7	$\frac{z - 4e}{2} = 5x - 15e$
7×2	8	$z - 4e = 10x - 30e$
$8 + 30e$	9	$z + 26e = 10x$
$9 - z$	10	$26e = 10x - z$
$10 \div 26$	11	$e = \frac{10x - z}{26} = 75$, the Miles travelled by B.
Then by the 4th Equation	12	$a = 5x - 15e = 120$, the Miles travelled by A.

PROOF.

$$\frac{a}{5} + 3e = 249$$

$$2a + 4e = 540$$

The

The Learner being now a little converſant with theſe Kind of Queſtions, let the laſt be repeated, and put Letters for all the Numbers both known and unknown, and if he finds any Difficulty in ſolving it, by comparing the two Operations, the former may in ſome Manner explain this; and to illuſtrate it the more, I have placed the Equations in the laſt Work, againſt their correſpondent Equations in the next Operation.

Queſtion 48. *Two Travellers,* A *and* B, *meeting on the Road, found, that if the Number of Miles travelled by* A *was divided by* 5, *and adding to the Quotient* 3 *times the Miles travelled by* B, *the Sum was* 249:

But the Miles travelled by A *being multiplied by* 2, *and added to* 4 *times the Miles travelled by* B, *the Sum was* 540. *How many Miles had each travelled?*

Let $a =$ the Number of Miles travelled by A, $e =$ the Number of Miles travelled by B, $x = 249$, $z = 540$, as before, but now put $d = 5$, $m = 3$, $q = 2$, $p = 4$.

By the firſt Condition	1	$\frac{a}{d} + me = x$, that is, $\frac{a}{5} + 3e = x$
By the ſecond Condition	2	$qa + pe = z$, that is, $2a + 4e = z$
$1 \times d$	3	$a + dme = dx$, that is, $a + 15e = 5x$
$3 - dme$	4	$a = dx - dme$, that is, $a = 5x - 15e$

Having found the Value of a from the firſt Equation, find its Value from the ſecond Equation.

$2 - pe$	5	$qa = z - pe$, that is, $2a = z - 4e$
$5 \div q$	6	$a = \frac{z - pe}{q}$, that is, $a = \frac{z - 4e}{2}$

Now make the fourth and ſixth Equations equal to one another, for they are both equal to the ſame Quantity a, which exterminates that unknown Quantity.

$4 \cdot 6$	7	$\frac{z - pe}{q} = dx - dme$, that is, $\frac{z - 4e}{2} = 5x - 15e$
$7 \times q$	8	$z - pe = dqx - dmeq$, that is, $z - 4e = 10x - 30e$

$8 + dmeq$

$8 + dmeq$	9	$dmeq + z - pe = dqx$, that is, $26e + z = 10x$
$9 - z$	11	$dmeq - pe = dqx - z$, that is, $26e = 10x - z$.

The unknown Quantity e being in two Terms, therefore divide by both the Co-efficients of e, as at Art. 52.

$10 \div dmq - p$	11	$e = \dfrac{dqx - z}{dmq - p} = 75$, that is, $e = \dfrac{10x - z}{26} = 75$, the Miles travelled by B.

And it being found that e is 75, we may find a by the fourth or sixth Equation to be 120.

And now for the future we shall put Letters for the Numbers that are known, as well as for those that are unknown.

Question 49. *There are two Armies ready to engage; if the Number of Soldiers in both Armies are added together, and that Sum multiplied by* 4, *the Product is* 84440:

But if the Number of Men in the greatest Army be multiplied by 2, *and added to the Product of the Number of Men in the lesser Army multiplied by* 3, *the Sum is* 52219. *To find the Number of Men in each Army?*

Let $a =$ the Number of Men in the greatest Army, $e =$ the Number of Men in the lesser Army, $d = 4$, $m = 84440$, $z = 2$, $x = 3$, $b = 52219$.

By the first Condition	1	$da + de = m$
By the second Condition	2	$za + xe = b$

Find the Value of a, in the first Equation.

$1 - de$	3	$da = m - de$
$3 \div d$	4	$a = \dfrac{m - de}{d}$

Now find the Value of a from the second Equation.

$2 - xe$	5	$za = b - xe$
$5 \div z$	6	$a = \dfrac{b - xe}{z}$

Make the fourth and ſixth Equations equal to one another to exterminate a.

$4 . 6$	7	$\dfrac{m - de}{d} = \dfrac{b - xe}{z}$
$7 \times d$	8	$m - de = \dfrac{db - dxe}{z}$
$8 \times z$	9	$zm - zde = db - dxe$

Now in this Equation e being on both Sides, find which of its *Co-efficients* dx or zd is the greateſt. zd is 8, but dx is 12, therefore tranſpoſe dxe, that the unknown Quantity, with the greateſt Co-efficient, may have the affirmative Sign, as at Art. 52.

$9 + dxe$	10	$dxe + zm - zde = db$
$11 - zm$	11	$dxe - zde = db - zm$
$11 \div dx - zd$	12	$e = \dfrac{db - zm}{dx - zd} = 9999$
By the ſixth Equation	13	$a = \dfrac{b - xe}{z} =$ 11111, the Number of Men in the greateſt Army.

Dividing the eleventh Equation by $dx - zd$, the two Co-efficients of e, as at Art. 52. gives the twelfth Equation.

PROOF.

$$4a + 4e = 84440$$
$$2a + 3e = 52219$$

Queſtion 50. *A Gentleman bought a Pair of Horſes for his Coach, his Son having learnt* Algebra, *the Father propoſed for him to determine the Price of each Horſe from ſaying,*

That if the Pounds both Horſes coſt were multiplied by 4, *and this Product divided by* 8, *the Quotient was* 20:

But

But if the Pounds the best Horse cost were multiplied by 3, and this Product added to 5 times the Pounds the worst Horse cost, the Sum was 158 Pounds. Now what was the Price of each Horse?

Let $a =$ the Pounds the best Horse cost, $e =$ the Pounds the worst Horse cost, $b = 4$, $d = 8$, $m = 20$, $p = 3$, $x = 5$, $z = 158$.

By the first Condition	1	$\frac{ba + be}{d} = m$
By the second Condition	2	$pa + xe = z$
$1 \times d$	3	$ba + be = dm$
$3 - be$	4	$ba = dm - be$
$4 \div b$	5	$a = \frac{dm - be}{b}$
$2 - xe$	6	$pa = z - xe$
$6 \div p$	7	$a = \frac{z - xe}{p}$
		To exterminate a
$5 \cdot 7$	8	$\frac{dm - be}{b} = \frac{z - xe}{p}$
$8 \times p$	9	$\frac{pdm - pbe}{b} = z - xe$
$9 \times b$	10	$pdm - pbe = bz - bxe$
$10 + bxe$	11	$bxe + pdm - pbe = bz$
$11 - pdm$	12	$bxe - pbe = bz - pdm$
$12 \div bx - pb$	13	$e = \frac{bz - pdm}{bx - pb} = 19$ Pounds, the Price of the worst Horse.
By the seventh Equation	14	$a = \frac{z - xe}{p} = 21$ Pounds, the Price of the best Horse.

PROOF.

$$\frac{4a + 4e}{8} = 20.$$

$$3a + 5e = 158.$$

Question 51. *Two young Gentlemen, who had studied Numbers, not agreeing about their Age, referred the Dispute to their Father, who smiling told them, that if the Age of the eldest was divided by*

by 2, to which Quotient adding 4 times the Age of the youngeſt, and extracting the ſquare Root of this Sum, it will be 10:

But if the Age of the eldeſt was multiplied by 3, and added to the Age of the youngeſt multiplied by 5, this Sum will be 201. To find the Age of each Perſon?

Let $a=$ the Age of the elder, $e=$ the Age of the younger, $b=2$, $d=4$, $m=10$, $p=3$, $z=5$, $r=201$.

By the firſt Condition	1	$\sqrt{\frac{a}{b}+de}=m$
By the ſecond Condition	2	$pa+ze=r$

Becauſe in the firſt Equation, a the unknown Quantity, is under the radical Sign, therefore ſquare both Sides of the Equation, as at Art. 49. The 1 ⊕ 2 in the *Regiſter* ſignifies that the firſt Equation being involved or raiſed to the ſecond Power or Square makes the third Equation, for ⊕ is the Sign of Involution.

1 ⊕ 2	3	$\frac{a}{b}+de=mm$
$3-de$	4	$\frac{a}{b}=mm-de$
$4\times b$	5	$a=bmm-bde$
$2-ze$	6	$pa=r-ze$
$6\div p$	7	$a=\frac{r-ze}{p}$
		Now to exterminate a
$5\cdot 7$	8	$\frac{r-ze}{p}=bmm-bde$
$8\times p$	9	$r-ze=pbmm-pbde$
$9+pbde$	10	$pbde+r-ze=pbmm$
$10-r$	11	$pbde-ze=pbmm-r$
$11\div pbd-z$	12	$e=\frac{pbmm-r}{pbd-z}=21$, the Age of the (youngeſt.
By the ſeventh Step	13	$a=\frac{r-ze}{p}=32$, the Age of the (eldeſt.

PROOF.

PROOF.

$$\sqrt{\frac{a}{2}+4e}=10.$$

$$3a+5e=201.$$

Question 52. *Two Tradesmen,* A *and* B, *comparing their Gains, found, that if the Pounds gained by* A *were multiplied by* 2, *to which adding* 3 *times the Pounds gained by* B, *the square Root of this Sum was* 11 *Pounds:*

But if 6 *times the Pounds gained by* B, *were added to the Quotient of the Pounds gained by* A *divided by* 10, *this Sum was* 47 *Pounds. To find the Gains of each Tradesman?*

Let $a=$ the Pounds gained by A, $e=$ the Pounds gained by B, $b=2$, $d=3$, $n=11$, $p=6$, $z=10$, $x=47$.

By the first Condition	1	$\sqrt{ba+de}=n$
By the second Condition	2	$pe+\frac{a}{z}=x$

In the first Equation the unknown Quantity a being under the radical Sign, square both Sides of the Equation as in the last Question.

1 ⊕ 2	3	$ba+de=nn$
$3-de$	4	$ba=nn-de$
$4\div b$	5	$a=\frac{nn-de}{b}$
$2\times z$	6	$zpe+a=zx$
$6-zpe$	7	$a=zx-zpe$
$5\cdot 7$	8	$\frac{nn-de}{b}=zx-zpe$
$8\times b$	9	$nn-de=bzx-bzpe$
$9+bzpe$	10	$bzpe+nn-de=bzx$
$10-nn$	11	$bzpe-de=bzx-nn$

$11 \div bzp - d$	12	$e = \frac{bzx - nn}{bzp - d} = 7$ Pounds gained (by B.
By the feventh Step	13	$a = zx - zpe = 50$ Pounds gained (by A.

PROOF.

$$\sqrt{2a + 3e} = 11.$$

$$6e + \frac{a}{10} = 47.$$

Queftion 53. *Two Perfons,* A *and* B, *owe fuch a Sum of Money, that if the Pounds* A *owes are divided by* 5, *to which Quotient adding* 4 *times the Pounds* B *owes, and extract the fquare Root of this Sum, it will be* 6 *Pounds:*

But if from 3 *times the Pounds* A *owes, is fubftracted* 50 *times the Pounds* B *owes, and extract the fquare Root of this Remainder, it will be* 10 *Pounds. What did each Perfon owe?*

Let $a =$ the Pounds A owes, $e =$ the Pounds B owes, $m = 5$, $n = 4$, $d = 6$, $p = 3$, $x = 50$, $z = 10$.

By the firft Condition	1	$\sqrt{\frac{a}{m} + ne} = d$
By the fecond Condition	2	$\sqrt{pa - xe} = z$

To find the Value of a in the firft Equation, raife it to the fecond Power as in the laft Queftion.

1 ⓖ 2	3	$\frac{a}{m} + ne = dd$
$3 - ne$	4	$\frac{a}{m} = dd - ne$
$4 \times m$	5	$a = mdd - mne$

To find the Value of a in the fecond Equation, raife it to the fecond Power as before.

$2 \circledcirc 2$	6	$pa - xe = zz$
$6 + xe$	7	$pa = zz + xe$
$7 \div p$	8	$a = \frac{zz + xe}{p}$

Now make the fifth and eighth Equations equal to one another to exterminate a.

$5 \cdot 8$	9	$\frac{zz + xe}{p} = mdd - mne$
$9 \times p$	10	$zz + xe = pmdd - pmne$
$10 + pmne$	11	$pmne + zz + xe = pmdd$
$11 - zz$	12	$pmne + xe = pmdd - zz$
$12 \div pmn + x$	13	$e = \frac{pmdd - zz}{pmn + x} = 4$ Pounds, the Debt (of B.
Then by the eighth Step	14	$a = \frac{zz + xe}{p} = 100$ Pounds, the Debt (of A.

PROOF.

$$\sqrt{\frac{a}{5} + 4e} = 6.$$

$$\sqrt{3a - 50e} = 10.$$

Question 54. *Two Men,* A *and* B, *going to Market with Eggs, if the Number of Eggs that* A *had were multiplied by* 6, *to which adding* 100, *and dividing the Sum by the Number of Eggs that* B *had, the Quotient is* 16:

And if from 9 *times the Number of Eggs* A *had, is substracted* 4 *times the Number of Eggs* B *had, there remains* 350. *How many Eggs had each Person?*

Let $a =$ the Number of Eggs A had, $e =$ the Number of Eggs B had, $d = 6$, $m = 100$, $p = 16$, $b = 9$, $x = 4$, $z = 350$.

	1	$\frac{da + m}{e} = p$ By the Question.
	2	$ba - xe = z$
$1 \times e$	3	$da + m = pe$

$3 - m$	4	$da = pe - m$
$4 \div d$	5	$a = \frac{pe - m}{d}$
$2 + xe$	6	$ba = z + xe$
$6 \div b$	7	$a = \frac{z + xe}{b}$

Make the fifth and ſeventh Equations equal to one another to exterminate a.

$5 \cdot 7$	8	$\frac{pe - m}{d} = \frac{z + xe}{b}$
$8 \times d$	9	$pe - m = \frac{dz + dxe}{b}$
$9 \times b$	10	$bpe - bm = dz + dxe$
$10 - dxe$	11	$bpe - dxe - bm = dz$
$11 + bm$	12	$bpe - dxe = dz + bm$
$12 \div bp - dx$	13	$e = \frac{dz + bm}{bp - dx} = 25$, the Number (of Eggs B had.
By the ſeventh Step	14	$a = \frac{z + xe}{b} = 50$, the Number of (Eggs A had.

PROOF.

$$\frac{6a + 100}{e} = 16.$$

$$9a - 4e = 350.$$

Queſtion 55. *Two Perſons, A and B, loſing at the Gaming-Table, were aſked how much they loſt, to which A replied, that if the Number of Pounds I loſt be multiplied by 3, and add 100 to the Product, if this Sum is divided by the Number of Pounds B loſt, the ſquare Root of this Quotient will be 10 Pounds:*

But if the Pounds B loſt be multiplied by 250, from which Product ſubſtracting 600, and dividing the Remainder by the Pounds A loſt, the ſquare Root of this Quotient will be 2 Pounds. How much had each Perſon loſt?

Let

Let $a =$ the Pounds A lost, $e =$ the Pounds B lost, $d = 3$, $m = 100$, $n = 10$, $x = 250$, $z = 600$, $b = 2$.

$$\left.\begin{array}{c|l} 1 & \sqrt{\dfrac{da+m}{e}} = n \\ 2 & \sqrt{\dfrac{xe-z}{a}} = b \end{array}\right\} \text{By the Question.}$$

To find the Value of a in the first Equation, raise it to the second Power by Art. 49.

$$\begin{array}{r|c|l} 1 \circledast 2 & 3 & \dfrac{da+m}{e} = nn \\ 3 \times e & 4 & da+m = enn \\ 4 - m & 5 & da = enn - m \\ 5 \div d & 6 & a = \dfrac{enn-m}{d} \end{array}$$

To find the Value of a in the second Equation, raise it to the second Power by Art. 49.

$$\begin{array}{r|c|l} 2 \circledast 2 & 7 & \dfrac{xe-z}{a} = bb \\ 7 \times a & 8 & xe - z = abb \\ 8 \div bb & 9 & \dfrac{xe-z}{bb} = a \end{array}$$

Make the sixth and ninth Equations equal to one another, to exterminate a.

$$\begin{array}{r|c|l} 6 \,.\, 9 & 10 & \dfrac{enn-m}{d} = \dfrac{xe-z}{bb} \\ 10 \times d & 11 & enn - m = \dfrac{dxe-dz}{bb} \\ 11 \times bb & 12 & bbnne - bbm = dxe - dz \end{array}$$

Because dx, one Co-efficient of e, is greater than $bbnn$, the other Co-efficient of e, therefore transpose $bbnne$, by Art. 52.

$12 - bbnne$	13	$-bbm = dxe - dz - bbnne$
Or	14	$dxe - dz - bbnne = -bbm$
$14 + dz$	15	$dxe - bbnne = dz - bbm$
$15 \div dx - bbnn$	16	$e = \frac{dz - bbm}{dx - bbnn} = 4$, the Pounds B (loſt.
By the ninth Step	17	$a = \frac{xe - z}{bb} = 100$, the Pounds A (loſt.

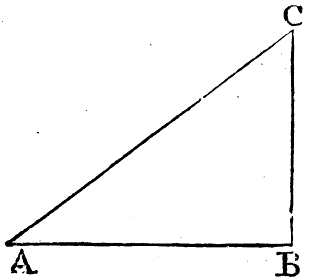
PROOF.

$$\sqrt{\frac{3a + 100}{e}} = 10.$$

$$\sqrt{\frac{250e - 600}{a}} = 2.$$

Queſtion 56. *In the right-angled Triangle* ABC, *there is given the Baſe* AB $= 4$, *and the Difference between the Hypothenuſe* AC *and Perpendicular* BC $= 2$. *To find the Hypothenuſe* AC *and Perpendicular* BC?

Let AC $= a$, BC $= e$, AB $= b = 4$, $m = 2$.



Having put Letters for the three Sides of the Triangle, and amongſt theſe there being two unknown Quantities a and e, therefore we muſt raiſe two Equations either from the Properties of the Figure, or from the Conditions of the Queſtion. And in the Solution of Geometrical Queſtions, I would recommend it to the Learner, that after all the Parts of the Figure which are neceſſary to the Solution of the Queſtion are expreſſed by Letters, to obſerve how many of them are unknown, for generally ſo many different Equations are raiſed from the Properties of the Figure, or the Conditions of the Queſtion; afterwards the Work is regulated by the Rules already given.

Now from the Property of the Figure, the Square of the Hypothenuſe AC, or aa, is equal to the Square of the Baſe AB, or bb, added to the Square of the Perpendicular BC, or ee, by 47 e 1.

That

That is	1	$aa = bb + ee$ from the Property of the Figure by 47 *e* 1.

Because by the Question, the Difference between the Hypothenuse AC, or a, and Perpendicular BC, or e, is $= 2$, or m.

Hence	2	$a - e = m$ by the Conditions of the Question.

Having raised the two Equations, proceed as in the former Examples, that is, first find the Value of a in the first Equation, by the Extraction of Roots, as at Art. 50.

1 ɯ 2	3	$a = \sqrt{bb + ee}$

Now find the Value of a, in the second Equation.

$2 + e$	4	$a = m + e$

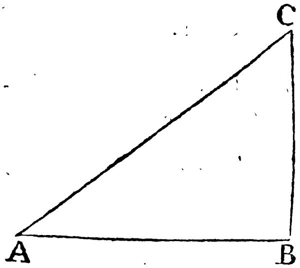
Make the third and fourth Equations equal to one another, to exterminate a.

3 · 4	5	$m + e = \sqrt{bb + ee}$

Because e the unknown Quantity is under the radical Sign, and there being no other Quantities on that Side of the Equation, but what are under the radical Sign, therefore square both Sides of the Equation, as at Art. 49.

5 ⓦ 2	6	$mm + 2mee + e = bb + ee$
$6 - ee$	7	$mm + 2me = bb$
$7 - mm$	8	$2me = bb - mm$
$8 \div 2m$	9	$e = \dfrac{bb - mm}{2m} = 3$, the Perpendicular BC.
By the fourth Step	10	$a = m + e = 5$, the Hypothenuse AC.

To prove these are the three Sides of a right-angled Triangle, square the Hypothenuse 5, and that will be equal to the Square of the Base 4, added to the Square of the Perpendicular 3; for this is the celebrated Property of the right-angled Triangle to have the Square of the Hypothenuse equal to the Sum of the Squares of the Base and Perpendicular.



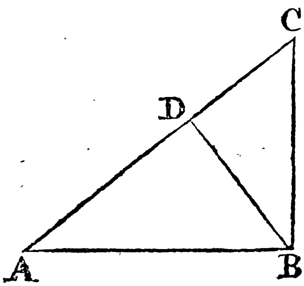
Question 57. *In the right-angled Triangle* ABC, *given the Perpendicular* BC $= 3$, *and the Difference between the Hypothenuse* AC, *and Base* AB $= 1$. *To find the Hypothenuse* AC, *and Base* BA?

Let AC $= a$, BC $= 3 = b$, AB $= e$, $x = 1$.

Then	1	$aa = bb + ee$, by the Property of the Figure, as in the last Question.
And	2	$a - e = x$ by the Question.

There being as many Equations raised from the Property of the Figure, and the Conditions of the Question, as there are unknown Quantities, the Work proceeds upon the same general Rules, thus

1 ω 2	3	$a = \sqrt{bb + ee}$
$2 + e$	4	$a = x + e$
$3 \cdot 4$	5	$x + e = \sqrt{bb + ee}$
5 ⓖ 2	6	$xx + 2xe + ee = bb + ee$
$6 - ee$	7	$xx + 2xe = bb$
$7 - xx$	8	$2xe = bb - xx$
$8 \div 2x$	9	$e = \dfrac{bb - xx}{2x} = 4$, the Base AB.
By the fourth Step	10	$a = x + e = 5$, the Hypothenuse AC.



Question 58. *In the right-angled Triangle* ABC, *there is given the Hypothenuse* AC $= 5$, *the Base* AB $= 4$, *and the Perpendicular* BC$=3$, *to find the Perpendicular* BD, *let fall from the Angle* B, *upon the Hypothenuse* AC.

Let AC $= b = 5$, AB $= m = 4$, BC $= x = 3$, DC $= a$, AD $= e$.

The

The Question requiring that we find BD, if we find CD we can answer the Question, for the Triangle BDC being a right-angled Triangle, BD being perpendicular to AC, consequently BC being known, and by finding DC, we shall afterwards easily find DB, by the common Property of the Triangle.

It is exactly the same, if we find AD, for the Triangle ADB is right-angled, and AB is given by the Question.

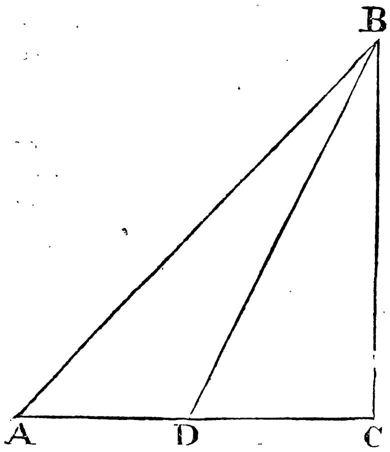
Now BD being a Perpendicular common to the two Triangles ABD, and BDC, let BD $= p$, then from the right-angled Triangle ABD, we have $mm - ee = pp$, and by the right-angled Triangle CBD, we have $xx - aa = pp$, from the same Reasoning as in the two last Questions.

Consequently	1	$mm - ee = xx - aa$, for both $mm - ee$ and $xx - aa$, are equal to the same Quantity pp, and therefore equal to one another.
And	2	$a + e = b$, that is, AD + DC = AC by the Figure.
		To find the Value of a in the first Equation.
$1 + aa$	3	$aa + mm - ee = xx$
$3 + ee$	4	$aa + mm = xx + ee$
$4 - mm$	5	$aa = xx + ee - mm$
5 ω 2	6	$a = \sqrt{xx + ee - mm}$
		Now find the Value of a in the second Equation.
$2 - e$	7	$a = b - e$
6 . 7	8	$\sqrt{xx + ee - mm} = b - e$
8 ☉ 2	9	$xx + ee - mm = bb - 2be + ee$
$9 - ee$	10	$xx - mm = bb - 2be$
$10 + 2be$	11	$2be + xx - mm = bb$
$11 + mm$	12	$2be + xx = bb + mm$
$12 - xx$	13	$2be = bb + mm - xx$
$13 \div 2b$	14	$e = \dfrac{bb + mm - xx}{2b} = 3.2 =$ AD.

Having found AD to be 3.2 it will be easy to find DB by what was said above. Thus,

16 the Square of AB.
— 10.24 the Square of AD.

5.76 (2.4 = DB, the Perpendicular required.
4
44) 176
176
0



Queſtion 59. *In the oblique Triangle* ADB, *there is given the Side* AB = 15, *the Side* BD = 12, *and the Side* AD = 6, *to find the Perpendicular* BC *falling without the Triangle from the Angle* B, *on the Side* AD, *continued.*

This Queſtion will be anſwered from finding DC, for the Triangle BCD being right-angled, and DB being known from finding DC, we may then find BC from the common Property of the Triangle DBC, as in the laſt Queſtion.

Let AB $= b = 15$, AD $= m = 6$, DB $= x = 12$, DC $= a$, then AC = AD + DC $= m + a$, BC $= e$.

Becauſe the Triangle ABC is right-angled, therefore if from the Square of AB, or bb, we ſubſtract the Square of AC, or $mm + 2ma + aa$, the Remainder is equal to the Square of CB, or ee.

Therefore | 1 | $bb - mm - 2ma - aa = ee$.

Becauſe the Triangle DBC is right-angled, by the ſame Reaſoning we have

Again | 2 | $xx - aa = ee$.

And

And as the first and second Equations are each $= ee$, therefore make them equal to one another, which exterminates every Power of e in those Equations.

$1 \cdot 2$	3	$xx - aa = bb - mm - 2ma - aa$
$3 + aa$	4	$xx = bb - mm - 2ma$
$4 + 2ma$	5	$2ma + xx = bb - mm$
$5 - xx$	6	$2ma = bb - mm - xx$
$6 \div 2m$	7	$a = \dfrac{bb - mm - xx}{2m} = 3.75 = DC.$

And from hence we may find BC as was said above, thus

$$\begin{array}{r} BD, \text{ or } x = 12 \\ 12 \\ \hline 144 \end{array} \qquad \begin{array}{r} DC, \text{ or } a = 3.75 \\ 3.75 \\ \hline 1875 \\ 2625 \\ 1125 \\ \hline 14.0625 \end{array}$$

$$\begin{array}{l} 144. \\ 14.0625 \\ \hline 129.9375 \; (11.39 = BC, \text{ the Perpendicular required.} \\ 1 \\ \hline 21)\,29 \\ 21 \\ \hline 223)\,893 \\ 669 \\ \hline 2269)\,22475 \\ 20421 \\ \hline 2054 \end{array}$$

Of

Of Quadratic EQUATIONS.

56. WHEN all the known Quantities are on one Side of the Equation, and thoſe Quantities only on the other Side which have ſome Power of the unknown Quantity; then if the unknown Quantity appears to be to the *ſecond Power or Square* in one Term, and to the *firſt Power* only in another Term; or if in one Term, its Power or Heighth is *double* its Power or Heighth in another Term, and there is no *other* Power of the unknown Quantity in the Equation, theſe Equations are called *Quadratic*, as in the following Queſtions.

Queſtion 60. *Two Men had ſuch a Number of Shillings, that the leſſer being ſubſtracted from the greater, there remains* 10:

But the Number of Shillings one Man had multiplied by the Number of Shillings the other Man had, the Product is 75. *To find each Man's Number of Shillings?*

Let $a =$ the greater Number of Shillings one of the Men had, $e =$ the leſſer Number of Shillings the other Man had, $b = 10$, $m = 75$.

Then	1	$a - e = b$ } By the Queſtion.
And	2	$ae = m$
$1 + e$	3	$a = b + e$
$2 \div e$	4	$a = \frac{m}{e}$
$3 \cdot 4$	5	$e + b = \frac{m}{e}$
$5 \times e$	6	$ee + be = m$

From comparing the ſixth Equation with what is ſaid above, it appears to be *Quadratic*, for one Quantity is ee, or e to the ſecond Power, and in the other Quantity it is only e, or e to the firſt Power.

And

And to reſolve this Equation, take b the Co-efficient of e to the firſt Power, and divide it by 2, the Quotient is $\frac{b}{2}$, which ſquare or multiply by itſelf, and the Product is $\frac{bb}{4}$, which add to both Sides of the Equation, thus

$$6c\,\square \quad \Big| \; 7 \; \Big| \; ee + be + \frac{bb}{4} = m + \frac{bb}{4}$$

The $c\,\square$ in the *Regiſter* ſignifies, that the ſixth Step is made a Square at the ſeventh Step, or the *Square is compleated.*

Now if we compare the Side of the Equation $ee + be + \frac{bb}{4}$, with ſome of the Examples at Art. 34. we ſhall find it to be a *rational* Quantity, or a Square, therefore extract the ſquare Root of both Sides of the Equation:

$$7\ \omega\ 2 \quad \Big| \; 8 \; \Big| \; e + \frac{b}{2} = \sqrt{m + \frac{bb}{4}}$$

$$8 - \frac{b}{2} \quad \Big| \; 9 \; \Big| \; e = \sqrt{m + \frac{bb}{4}} : - \frac{b}{2}$$

In Numbers.

$m = 75$

$\frac{bb}{4} = 25$

100 the ſquare Root of which is 10

$-5 = -\frac{b}{2}$

$5 = e$, the Number of Shillings one of the Men had.

Then by the fourth Step $a = \frac{m}{e} = 15$, the Number of Shillings the other Man had.

Z PROOF.

PROOF.

$$a - e = 10$$
$$ae = 75$$

Queſtion 61. *There are two Numbers, if the Square of the leſſer is taken from the greater, there remains* 36:

But the greater being added to 6 *times the leſſer, the Sum is* 148. *What are the two Numbers?*

Let $a =$ the greater Number, $e =$ the leſſer Number, $b = 36$, $m = 6$, $x = 148$.

Then	1	$a - ee = b$ } By the Queſtion.
And	2	$a + me = x$ }
$1 + ee$	3	$a = b + ee$
$2 - me$	4	$a = x - me$
$3 \cdot 4$	5	$b + ee = x - me$
$5 - b$	6	$ee = x - me - b$
$6 + me$	7	$ee + me = x - b$

The unknown Quantities being brought on one Side of the Equation, the Equation appears to be *Quadratic*, by Art. 56.

Now the Co-efficient of the firſt Power of e is m, which divided by 2 is $\frac{m}{2}$, this ſquared is $\frac{mm}{4}$, and adding $\frac{mm}{4}$ to both Sides of the Equation as in the laſt Queſtion, we have

$7\,c\,\square$	8	$ee + me + \frac{mm}{4} = x - b + \frac{mm}{4}$

The $7\,c\,\square$ ſignifies that the ſeventh Equation is made a compleat Square, at the eighth Step.

And extracting the Roots of both Sides of the Equation, as in the laſt Queſtion,

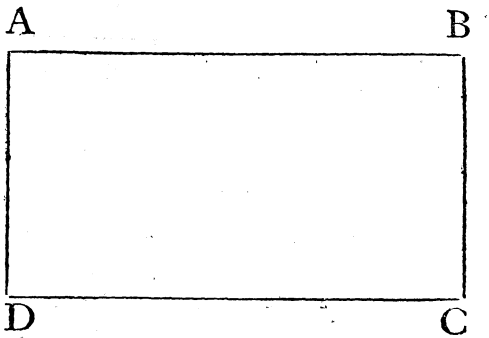
$8 \text{ w } 2$	9	$e + \frac{m}{2} = \sqrt{x - b + \frac{mm}{4}}$
$9 - \frac{m}{2}$	10	$e = \sqrt{x - b + \frac{mm}{4}} : - \frac{m}{2} = 8$, (the leſſer Number.
By the fourth Step	11	$a = x - me = 100$, the greater Num-(ber.

PROOF.

$$a - ee = 36$$
$$a + 6e = 148$$

Queſtion 62. *In the Parallelogram* ABCD, *if from the longeſt Side* AB *multiplied by* 3, *is ſubſtracted the Square of the ſhorteſt Side* BC, *the Remainder will be* 5:

But if the longeſt Side AB *is added to* 4 *times the ſhorteſt Side* BC, *the Sum is* 30. *To find the Sides of the Parallelogram* AB, *and* BC?



Let $a = AB$, $BC = e$, $d = 3$, $m = 5$, $z = 4$, $x = 30$.

	1	$da - ee = m$ } By the Queſtion.
	2	$a + ze = x$ }
$1 + ee$	3	$da = m + ee$
$3 \div d$	4	$a = \frac{m + ee}{d}$
$2 - ze$	5	$a = x - ze$

$4 \cdot 5$	6	$\frac{m + ee}{d} = x - ze$
$6 \times d$	7	$m + ee = dx - dze$
$7 - m$	8	$ee = dx - dze - m$
$8 + dze$	9	$ee + dze = dx - m$

Now the Equation appears to be *Quadratic* by Art. 56. and the Co-efficient of e is dz, which divided by 2, is $\frac{dz}{2}$, this ſquared is $\frac{ddzz}{4}$, which added to both Sides of the Equation, as in the two laſt Examples, we have

$9\,c\,\square$	10	$ee + dze + \frac{ddzz}{4} = dx - m + \frac{ddzz}{4}$

And extracting the Roots of both Sides of the Equation, as in the two laſt Queſtions,

10 ω 2	11	$e + \frac{dz}{2} = \sqrt{dx - m + \frac{ddzz}{4}}$
$11 - \frac{dz}{2}$	12	$e = \sqrt{dx - m + \frac{ddzz}{4}} : - \frac{dz}{2} = 5$ (= BC.
From the fifth Step	13	$a = x - ze = 10 =$ AB.

PROOF.

$3a - ee = 5$

$a + 4e = 30$

Queſtion 63. *Two Gentlemen having had their Parks ſurveyed, had loſt the Account, but remembered, that if the Number of Acres in* A's *Park was added to the Number of Acres in* B's *Park, the Sum was* 110:

But if the Number of Acres in B's *Park was multiplied by* 80, *from which Product ſubſtracting the Square of the Number of Acres in* A's *Park, there remained* 400. *How many Acres was there in each Park?*

Let

Let $a =$ the Number of Acres in A's Park, $e =$ the Number of Acres in B's Park, $b = 110$, $m = 80$, $x = 400$.

	1	$a + e = b$ } By the Queſtion.
	2	$me - aa = x$ }
$1 - a$	3	$e = b - a$
$2 + aa$	4	$me = x + aa$
$4 \div m$	5	$e = \frac{x + aa}{m}$
$3 \cdot 5$	6	$\frac{x + aa}{m} = b - a$
$6 \times m$	7	$x + aa = mb - ma$
$7 - x$	8	$aa = mb - ma - x$
$8 + ma$	9	$aa + ma = mb - x$

Here the Equation appears *Quadratic*, and compleating the Square as in the former Examples, we have

9 c □	10	$aa + ma + \frac{mm}{4} = mb - x + \frac{mm}{4}$

And extracting the ſquare Roots of both Sides of the Equation, as in the former Examples,

10 ω 2	11	$a + \frac{m}{2} = \sqrt{mb - x + \frac{mm}{4}}$
$11 - \frac{m}{2}$	12	$a = \sqrt{mb - x + \frac{mm}{4}} : - \frac{m}{2} = 60$, (the Number of Acres in A's Park.
From the third Step	13	$e = b - a = 50$, the Number of Acres (in B's Park.

PROOF.

$$a + e = 110$$
$$80e - aa = 400$$

The

The Manner of ſubſtituting one Quantity for ſeveral others explained.

57. But if, after the Work is prepared for having the *Square compleated*, it appears that the firſt Power of the unknown Quantity is in more Terms than one, it will be more convenient to *ſubſtitute* ſome other Letter, for the Co-efficients of the firſt Power of the unknown Quantity, as in the following Examples.

Queſtion 64. *A Gentleman propoſed to give his two Sons*, A *and* B, *each an Eſtate, on the Condition, they could tell him what were their Rents, by knowing, that if the Square of the Rent of the Eſtate he intended to give* A *was added to the ſame Rent multiplied by* 7, *and the Sum added to the Rent of the Eſtate he intended to give* B, *when multiplied by* 4, *this Sum would be* 4220 *Pounds:*

But if the Sum of the Rents of the two Eſtates was divided by 10, *the Quotient was* 11 *Pounds. What was the Rent of each Eſtate?*

Let $a =$ the Rent of the Eſtate A was to have, $e =$ the Rent of the Eſtate B was to have, $b = 7$, $m = 4$, $d = 4220$, $p = 10$, $x = 11$.

$$\left.\begin{array}{c|l} 1 & aa + ba + me = d \\ 2 & \dfrac{a+e}{p} = x \end{array}\right\}\text{By the Queſtion.}$$

Theſe being the two Equations which ariſe from the Queſtion, and becauſe the Terms are more ſimple that have the unknown Quantity e, than thoſe that have the unknown Quantity a, it may be more convenient to find the Value of e, in each of the two given Equations. This Caution the Learner may obſerve for the future, to find the Value of that unknown Quantity whoſe Terms are the moſt ſimple in the given Equations; and thoſe may be taken for the more ſimple, whoſe Powers are the loweſt in both the Equations that ariſe from the Queſtion; thus, if one of the unknown Quantities is only to the *firſt* Power in both the given Equations, when the other unknown Quantity is to the *ſecond* Power in one of the given Equations, the Terms of the former may be ſaid to be more ſimple, and there-

therefore beſt to find the Value of that unknown Quantity: The Reader will find this Method obſerved in the following Queſtions, and comparing their Work with what is ſaid may make this Direction more intelligible.

$1 - ba$	3	$aa + me = d - ba$
$3 - aa$	4	$me = d - ba - aa$
$4 \div m$	5	$e = \dfrac{d - ba - aa}{m}$
$2 \times p$	6	$a + e = px$
$6 - a$	7	$e = px - a$
$5 \cdot 7$	8	$px - a = \dfrac{d - ba - aa}{m}$
$8 \times m$	9	$mpx - ma = d - ba - aa$
$9 + aa$	10	$aa + mpx - ma = d - ba$
$10 + ba$	11	$aa + ba + mpx - ma = d$
$11 - mpx$	12	$aa + ba - ma = d - mpx$

Here the Equation appears to be Quadratic, and the firſt Power of a is in two Terms, *viz.* ba and ma, the two Co-efficients being b and m, and connected by the Sign —.

But b and m, being known Quantities, therefore $b - m = 7 - 4 = 3$, now *ſubſtitute*, or put $z = 3$, or $z = b - m$, then the laſt Equation is,

By Subſtitution | 13 | $aa + za = d - mpx$; for by Subſtitution $za = ba - ma$, and therefore in the room of $ba - ma$, we uſe only za. Now taking z for the Co-efficient of a, and compleating the Square as before,

13 c $\square$	14	$aa + za + \dfrac{zz}{4} = d - mpx + \dfrac{zz}{4}$
14 ɯ 2	15	$a + \dfrac{z}{2} = \sqrt{d - mpx + \dfrac{zz}{4}}$
$15 - \dfrac{z}{2}$	16	$a = \sqrt{d - mpx + \dfrac{zz}{4}} : - \dfrac{z}{2} = 60$, the Rent of the Eſtate which A was to have.
By the ſeventh Step	17	$e = px - a = 50$, the Rent of the (Eſtate which B was to have.

PROOF.

PROOF.

$$aa+7a+4e=4220.$$
$$\frac{a+e}{10}=11.$$

It may be just observed to the Learner, that the Method of *Substitution* is only to save Trouble and Labour, for after the twelfth Step, if we had not substituted $b-m=z$, then to have *compleated the Square*, we must have divided $b-m$ the two Co-efficients of a by 2, the Quotient of which is $\frac{b-m}{2}$, which squared is $\frac{bb-2bm+mm}{4}$, and this must have been added to both Sides of the Equation, whereas by *substituting* $b-m=z$, the Quantity to be added on both Sides of the Equation is only $\frac{zz}{4}$.

Question 65. *A Draper bought a Parcel of Linen, and a Parcel of Woollen Cloth, if the Square of the Pounds he gave for the Linen Cloth be divided by* 4, *and to this Quotient there is added the Pounds each Sort cost, the Sum is* 1000 *Pounds:*

But if the Pounds the Linen cost is added to the Quotient of the Pounds the Woollen cost, divided by 8, *the Sum is* 65 *Pounds. How much was given for each Sort?*

Let $a=$ the Pounds the Linen cost, $e=$ the Pounds the Woollen cost, $b=4$, $d=1000$, $m=8$, $x=65$.

$$\begin{array}{r|c|l}
 & 1 & \frac{aa}{b}+a+e=d \\
 & 2 & a+\frac{e}{m}=x \\
\hline
1-a & 3 & \frac{aa}{b}+e=d-a \\
3-\frac{aa}{b} & 4 & e=d-a-\frac{aa}{b}
\end{array}$$

(Steps 1 and 2:) By the Question.

$2 \times m$	5	$ma + e = mx$
$5 - ma$	6	$e = mx - ma$
$4 \cdot 6$	7	$mx - ma = d - a - \frac{aa}{b}$
$7 + \frac{aa}{b}$	8	$\frac{aa}{b} + mx - ma = d - a$
$3 + a$	9	$\frac{aa}{b} + a + mx - ma = d$
$9 - mx$	10	$\frac{aa}{b} + a - ma = d - mx$
$10 \times b$	11	$aa + ba - bma = bd - bmx$

Here the Equation appears *Quadratic*, and the firſt Power of the unknown Quantity a, has two Co-efficients b and bm, both which are known, but $b - bm = 4 - 32 = -28$, therefore as -28 is a *negative* Quantity, ſubſtitute $-z = -28$, or $-z = b - m$, then the laſt Equation becomes,

By Subſtitution | 12 | $aa - za = bd - bmx$, for $ba - bma$ is a *negative* Quantity, bm being greater than b: And compleating the Square as before,

$12\ c\ \square$	13	$aa - za + \frac{zz}{4} = bd - bmx + \frac{zz}{4}$,
		for $-\frac{z}{2} \times -\frac{z}{2} = +\frac{zz}{4}$, by Art. 9.

And extracting the ſquare Root as in the former Queſtions,

$13\ ɯ\ 2$	14	$a - \frac{z}{2} = \sqrt{bd - bmx + \frac{zz}{4}}$
$14 + \frac{z}{2}$	15	$a = \sqrt{bd - bmx + \frac{zz}{4}} : + \frac{z}{2} = 60$ (Pounds, the Linen coſt.
By the ſixth Equation,	16	$e = mx - ma = 40$ Pounds, the (Woollen coſt,

 PROOF.

PROOF.

$$\frac{aa}{4} + a + e = 1000.$$

$$a + \frac{e}{8} = 65.$$

To resolve a Quadratic Equation when the Square of the unknown Quantity has a Co-efficient.

58. But if the Square of the unknown Quantity has any Co-efficient besides *Unity*, or 1, then before you begin to *compleat the Square*, divide every Term in the Equation by that Co-efficient, after which compleat the Square, and proceed as before.

Question 66. *To find two Numbers, that the Square of the greater being multiplied by* 4, *if this Product is added to* 3 *times the lesser, the Sum shall be* 1606:

But if 5 *times the greater is added to* 6 *times the lesser, the Sum shall be* 112.

Let a = the greater Number, e = the lesser Number, $b = 4$, $d = 3$, $m = 1606$, $p = 5$, $x = 6$, $z = 112$.

	1	$baa + de = m$	By the Question.
	2	$pa + xe = z$	
$1 - baa$	3	$de = m - baa$	
$3 \div d$	4	$e = \frac{m - baa}{d}$	
$2 - pa$	5	$xe = z - pa$	
$5 \div x$	6	$e = \frac{z - pa}{x}$	
$4 . 6$	7	$\frac{z - pa}{x} = \frac{m - baa}{d}$	
$7 \times x$	8	$z - pa = \frac{xm - xbaa}{d}$	

8 × d

$8 \times d$	9	$dz - dpa = xm - xbaa$
$9 + xbaa$	10	$xbaa + dz - dpa = xm$
$10 - dz$	11	$xbaa - dpa = xm - dz$

The Equation appearing to be *Quadratic*, and all the known Quantities, except thoſe which contain the unknown one, being on one Side of the Equation, and the higheſt Power of the unknown Quantity having a Co-efficient, divide by that *Co-efficient*.

$11 \div xb$	12	$aa - \frac{dpa}{xb} = \frac{xm - dz}{xb}$

To avoid the Trouble of dividing $\frac{dp}{xb}$, the Co-efficient of a, by 2, and ſquaring the Quotient, and adding it to both Sides of the Equation to compleat the Square, as in the former Queſtions, ſubſtitute $-r = -\frac{dp}{xb} = -.625$ then,

By Subſtitution	13	$aa - ra = \frac{xm - dz}{xb}$
13 c □	14	$aa - ra + \frac{rr}{4} = \frac{xm - dz}{xb} + \frac{rr}{4}$

Now extracting the ſquare Root as in the laſt Queſtion,

14 ɯ 2	15	$a - \frac{r}{2} = \sqrt{\frac{xm - dz}{xb} + \frac{rr}{4}}$
$15 + \frac{r}{2}$	16	$a = \sqrt{\frac{xm - dz}{xb} + \frac{rr}{4}} : + \frac{r}{2} = 20$, (the greater Number.
By the ſixth Step	17	$e = \frac{z - pa}{x} = 2$, the leſſer Number.

PROOF.

$4\ aa + 3e = 1606.$
$5\ a + 6e = 112.$

Queſtion 67. *Two Gameſters,* A *and* B, *loſing at the Gaming-Tables, upon comparing their Loſſes, found, that if the Square of the Pounds* A *loſt was multiplied by* 5, *and this Product added to* 6 *times the Pounds* B *loſt, the Sum was* 548 *Pounds:*

But if the Pounds A *loſt was multiplied by* 3, *and to this Product adding the Pounds* B *loſt multiplied by* 2, *the Sum was* 46 *Pounds. To find the Loſs of each?*

Let $a =$ the Pounds A loſt, $e =$ the Pounds B loſt, $x = 5$, $m = 6$, $d = 548$, $b = 3$, $z = 2$, $r = 46$.

	1	$xaa + me = d$ } By the Queſtion.
	2	$ba + ze = r$
$1 - xaa$	3	$me = d - xaa$
$3 \div m$	4	$e = \dfrac{d - xaa}{m}$
$2 - ba$	5	$ze = r - ba$
$5 \div z$	6	$e = \dfrac{r - ba}{z}$
$4 \cdot 6$	7	$\dfrac{r - ba}{z} = \dfrac{d - xaa}{m}$
$7 \times m$	8	$\dfrac{rm - mba}{z} = d - xaa$
$8 \times z$	9	$rm - mba = zd - zxaa$
$9 + zxaa$	10	$zxaa + rm - mba = zd$
$10 - rm$	11	$zxaa - mba = zd - rm$

The Equation being *Quadratic,* and all thoſe Terms which contain any Power of a being on one Side of the Equation, divide by the Co-efficient of its higheſt Power.

$11 \div zx$	12	$aa - \dfrac{mba}{zx} = \dfrac{zd - rm}{zx}$
		Subſtitute $-p = -\dfrac{mb}{zx} = -1.8$
By Subſtitution	13	$aa - pa = \dfrac{zd - rm}{zx}$
$13 c \square$	14	$aa - pa + \dfrac{pp}{4} = \dfrac{zd - rm}{zx} + \dfrac{pp}{4}$

14 ω 2

14 ω 2	15	$a - \frac{p}{2} = \sqrt{\frac{zd - rm}{zx} + \frac{pp}{4}}$
$15 + \frac{p}{2}$	16	$a = \sqrt{\frac{zd - rm}{zx} + \frac{pp}{4}} : + \frac{p}{2} = 10$ (Pounds, the Sum loſt by A.
By the ſixth Step	17	$e = \frac{r - ba}{z} = 8$ Pounds, the Sum (loſt by B.

PROOF.

$$5aa + 6e = 548$$
$$3a + 2e = 46$$

Queſtion 68. *Two Brothers,* A *and* B, *trying each other's Skill in Algebra, ſays the eldeſt Brother, the Sum of our Ages is* 45:
But, ſays the youngeſt, if they are multiplied together, the Product is 500. *What is the Age of each of them?*

Let $a =$ the Age of the eldeſt, $e =$ the Age of the youngeſt, $s = 45$, $p = 500$.

	1	$a + e = s$ } By the Queſtion.
	2	$ae = p$
$1 - e$	3	$a = s - e$
$2 \div e$	4	$a = \frac{p}{e}$
3 · 4	5	$\frac{p}{e} = s - e$
$5 \times e$	6	$p = se - ee$

Becauſe the Square of the unknown Quantity has the Sign —, therefore tranſpoſe it, that the higheſt Power of the unknown Quantity may have the *affirmative* Sign.

$6 + ee$	7	$ee + p = se$
$7 - p$	8	$ee = se - p$
$8 - se$	9	$ee - se = -p$
9 c □	10	$ee - se + \frac{ss}{4} = \frac{ss}{4} - p$

Queſtion 67. *Two Gameſters,* A *and* B, *loſing at the Gaming-Tables, upon comparing their Loſſes, found, that if the Square of the Pounds* A *loſt was multiplied by* 5, *and this Product added to* 6 *times the Pounds* B *loſt, the Sum was* 548 *Pounds:*

But if the Pounds A *loſt was multiplied by* 3, *and to this Product adding the Pounds* B *loſt multiplied by* 2, *the Sum was* 46 *Pounds. To find the Loſs of each?*

Let $a =$ the Pounds A loſt, $e =$ the Pounds B loſt, $x = 5$, $m = 6$, $d = 548$, $b = 3$, $z = 2$, $r = 46$.

	1	$xaa + me = d$ } By the Queſtion.
	2	$ba + ze = r$
$1 - xaa$	3	$me = d - xaa$
$3 \div m$	4	$e = \dfrac{d - xaa}{m}$
$2 - ba$	5	$ze = r - ba$
$5 \div z$	6	$e = \dfrac{r - ba}{z}$
$4 \cdot 6$	7	$\dfrac{r - ba}{z} = \dfrac{d - xaa}{m}$
$7 \times m$	8	$\dfrac{rm - mba}{z} = d - xaa$
$8 \times z$	9	$rm - mba = zd - zxaa$
$9 + zxaa$	10	$zxaa + rm - mba = zd$
$10 - rm$	11	$zxaa - mba = zd - rm$

The Equation being *Quadratic,* and all thoſe Terms which contain any Power of a being on one Side of the Equation, divide by the Co-efficient of its higheſt Power.

$11 \div zx$	12	$aa - \dfrac{mba}{zx} = \dfrac{zd - rm}{zx}$
		Subſtitute $-p = -\dfrac{mb}{zx} = -1.8$
By Subſtitution	13	$aa - pa = \dfrac{zd - rm}{zx}$
$13\ c\ \square$	14	$aa - pa + \dfrac{pp}{4} = \dfrac{zd - rm}{zx} + \dfrac{pp}{4}$

14 *w* 2

$14 \text{ ш } 2$	15	$a - \frac{p}{2} = \sqrt{\frac{zd - rm}{zx} + \frac{pp}{4}}$
$15 + \frac{p}{2}$	16	$a = \sqrt{\frac{zd - rm}{zx} + \frac{pp}{4}} : + \frac{p}{2} = 10$ (Pounds, the Sum lost by A.
By the sixth Step	17	$e = \frac{r - ba}{z} = 8$ Pounds, the Sum (lost by B.

PROOF.

$$5aa + 6e = 548$$
$$3a + 2e = 46$$

Question 68. *Two Brothers,* A *and* B, *trying each other's Skill in Algebra, says the eldest Brother, the Sum of our Ages is* 45:
But, says the youngest, if they are multiplied together, the Product is 500. *What is the Age of each of them?*

Let $a =$ the Age of the eldest, $e =$ the Age of the youngest, $s = 45$, $p = 500$.

	1	$a + e = s$ } By the Question.
	2	$ae = p$
$1 - e$	3	$a = s - e$
$2 \div e$	4	$a = \frac{p}{e}$
$3 \cdot 4$	5	$\frac{p}{e} = s - e$
$5 \times e$	6	$p = se - ee$

Because the Square of the unknown Quantity has the Sign —, therefore transpose it, that the highest Power of the unknown Quantity may have the *affirmative* Sign.

$6 + ee$	7	$ee + p = se$
$7 - p$	8	$ee = se - p$
$8 - se$	9	$ee - se = -p$
$9 \text{ c } \square$	10	$ee - se + \frac{ss}{4} = \frac{ss}{4} - p$

$10 \, \omega 2$	11	$e - \frac{s}{2} = \sqrt{\frac{ss}{4} - p}$
$11 + \frac{s}{2}$	12	$e = \frac{s}{2} + \sqrt{\frac{ss}{4} - p} = 25$, the Age (of the youngeſt.
By the third Step	13	$a = s - e = 20$, the Age of the eldeſt.

This Anſwer to the Queſtion contains an *Abſurdity*, for e that is put for the Age of the *youngeſt* Brother is 25, when a that is put for the Age of the *eldeſt* Brother is only 20.

The two Roots of Quadratic Equations explained.

59. And now we ſhall explain to the young *Analyſt*, that in every *Quadratic* Equation, the unknown Quantity has two Values or Roots, ſometimes one is *affirmative*, and the other *negative*, and ſometimes both are *affirmative*.

There are *three Forms* of Quadratic Equations.

The firſt is the ſixth Step of Queſtion 60, where we have $ee + be = m$.

And of this *Form* are the Equations at Queſtion 61, Step 7. Queſtion 62, Step 9. Queſtion 63, Step 9. Queſtion 64, Step 12.

The ſecond *Form* is the twelfth Step of Queſtion 65, where we have $aa - za = bd - bmx$.

And of this *Form* are the Equations at Queſtion 66, Step 11. Queſtion 67, Step 13.

The Difference between theſe two *Forms* of Quadratic Equations, is only in the loweſt Power of the unknown Quantity having the Sign $+$ or $-$, for in the firſt *Form* it has the Sign $+$, it being be, but in the ſecond *Form* it has the Sign $-$, for it is $-za$. And if the loweſt Power of the unknown Quantity has ſeveral Co-efficients connected by the Signs $+$ or $-$, as at Queſtion 64, Step 12. Queſtion 65, Step 11. Then if the Sum of the *poſitive* or *affirmative* Co-efficients exceeds the Sum of the *negative* Co-efficients, the Equation is of the firſt *Form*: But, on the contrary, if the Sum of the *negative* Co-efficients exceeds the Sum of the *poſitive* or *affirmative* Co-efficients, then the Equation is of the ſecond *Form*.

But the third *Form* is the ninth Step of the laſt Queſtion, where we have $ee - se = -p$, which differs from the other two *Forms* of Quadratic Equations, in this, that if the Side of the Equation, which is known, conſiſts but of one Quantity, as in the preſent Caſe, it has the Sign —; and if that Side of the Equation conſiſts of ſeveral known Quantities connected by the Signs + or —, that then the Sum of the *negative* Quantities is always greater than the Sum of the *affirmative* Quantities; but in the firſt and ſecond *Form*, if there is but one known Quantity, which compoſes that Side of the Equation, it will always have the *affirmative* Sign; and if there are ſeveral known Quantities connected by the Signs + or —, that then the Sum of the *affirmative* will always exceed the Sum of the *negative* Quantities.

Now of the two Values or Roots of a in the firſt and ſecond *Form* of Quadratic Equations, one is *affirmative*, and the other *negative*; and as the negative Value in theſe Equations does not come out in the Operation without a Miſtake in the Work, therefore theſe two Forms of Quadratic Equations give the true Numbers required.

But the *two* Values or Roots of a in the third *Form* are both *affirmative*, and the Anſwer ſometimes giving one, and ſometimes the other Root, and it being doubtful in many Caſes which of theſe two Values of a will anſwer the Conditions of the Queſtion; this *Form* of Quadratic Equations is therefore called the *Ambiguous* Form.

Before we ſhow the Reaſon of theſe two Values or Roots of the unknown Quantity in Quadratic Equations, and how from having found one Number, or Value, the Learner may find the other Number; we ſhall explain the Diviſion in *Algebra*, where the Quotient conſiſts of ſeveral Quantities connected by the Signs + and —.

60. *The Nature of Diviſion explained, when the Quotient conſiſts of ſeveral Quantities connected by the Signs + or —.*

To render this the eaſier to the Learner, let us reſume Example 1, Article 22, where we are to divide $ab + am$ by a, which being placed as uſual in common Arithmetic, thus,

Now the Number of times a may be had in $a b$ is b, that is, b is the Quotient of $a b$ divided by a; place b in the Quotient, multiply it by a, and place the Product $a b$ as in common Division, and subtracting it from $a b + a m$ the Dividend, there remains $a m$; then find how many times a will go in $a m$, and it is m, that is, m is the Quotient of $a m$ divided by a, and because the Signs of the Divisor a, and Dividend $a m$ are alike, therefore it must be $+ m$, which being placed in the Quotient, and multiplied by a, the Product is $a m$, which placing under $a m$, and subtracting it from $a m$, there remains o.

$$\begin{array}{l} a)\ a b + am\ (b + m \\ \quad a b \\ \hline \qquad + a m \\ \qquad + a m \\ \hline \qquad\quad 0 \end{array}$$

Hence the Quotient is $b + m$.

To divide $x x + x m + x a b$ by x.

$$\begin{array}{l} x)\ x x + x m + x a b\ (x + m + a b \\ \quad x x \\ \hline \qquad x m + x a b \\ \qquad x m \\ \hline \qquad\qquad x a b \\ \qquad\qquad x a b \\ \hline \qquad\qquad\ 0 \end{array}$$

Here dividing $x x$ by x, the Quotient is x, which placed in the Quotient, and multiplied by the Divisor x, and placing the Product $x x$ under the Dividend, from which subtracting it, there remains $x m + x a b$.

Then dividing $x m$ by x, the Quotient is m, or $+ m$, for the Signs of $x m$ and x are alike, put $+ m$ in the Quotient, by which multiply the Divisor x, and put the Product $x m$ under $x m + x a b$, and subtracting, there remains $x a b$.

Then dividing $x a b$ by x, the Quotient is $a b$, or $+ a b$, for the Signs of $x a b$ and x are alike, put $+ a b$ in the Quotient, by which multiply the Divisor x, and put the Product $x a b$, under $x a b$, and subtracting, there remains o, hence the Quotient is $x + m + a b$.

To

To divide $xx+2xa+aa$ by $x+a$.

$$\begin{array}{l} x+a)\,xx+2xa+aa\,(x+a \\ \quad xx+xa \\ \hline \qquad xa+aa \\ \qquad xa+aa \\ \hline \qquad\quad 0 \end{array}$$

Dividing xx by x, the Quotient is x, by which multiplying the Divisor $x+a$, the Product is $xx+xa$, which placed under the Dividend, and subtracted, there remains $xa+aa$.

Then dividing xa by x, the Quotient is a, or $+a$, for the Signs of xa and a are alike, put $+a$ in the Quotient, multiplying it by the Divisor $x+a$, the Product is $xa+aa$, which put under the Remainder $xa+aa$, and subtracting, there remains 0, hence the Quotient is $x+a$.

To divide $aa-bb$ by $a+b$.

$$\begin{array}{l} a+b)\,aa-bb\,(a-b \\ \quad aa+ab \\ \hline \qquad -ab-bb \\ \qquad -ab-bb \\ \hline \qquad\quad 0 \end{array}$$

Dividing aa by a, the Quotient is a, and multiplying the Divisor by a, gives $aa+ab$, this subtracted from the Dividend leaves $-ab-bb$; for here the Quantity ab, which is to be subtracted, is, by the Rule for Subtraction, to have its Sign changed, and then added, hence $+ab$ becomes in the Remainder $-ab$.

Then dividing $-ab$ by a, the Quotient is $-b$, for the Signs of ab and a are now unlike; multiplying the Divisor $a+b$, by $-b$, and subtracting the Product $-ab-bb$, from the Remainder $-ab-bb$, there remains 0, hence the Quotient is $a-b$.

To divide $aaa-3aax+3axx-xxx$ by $a-x$.

$$\begin{array}{l} a-x)\,aaa-3aax+3axx-xxx\,(aa-2ax+xx \\ \quad aaa-aax \\ \hline \qquad -2aax+3axx-xxx \\ \qquad -2aax+2axx \\ \hline \qquad\qquad axx-xxx \\ \qquad\qquad axx-xxx \\ \hline \qquad\qquad\quad 0 \end{array}$$

In

In these Divisions we may at Pleasure take any Term in the Dividend we have a Mind to use first, and find how many times any Term in the Divisor can be had in it, and when the Divisor is multiplied by the Quotient Quantity, we substract it from the whole Dividend, that is, take any Term in the Product, from any Term in the Dividend, without regarding whether they stand immediately over one another or no.

And to discover how many times any one Quantity can be had in another, we are only to consider into what Quantities we must multiply that Term in the Divisor, to make it the same with the Term in the Dividend, at which we ask the Question. Or, it is no more than to find the Quotient, which arises from dividing that particular Quantity in the Dividend, by the Quantity in the Divisor, which is done by the Rules in Division. Let us take the last Example, and change the Position of the Quantities:

$$
\begin{array}{l}
-x+a)\ -xxx+aaa+3axx-3aax\ (xx+aa-2ax \\
\qquad\quad -xxx \qquad\quad +axx \\
\hline
\qquad\qquad\qquad aaa+2axx-3aax \\
\qquad\qquad\qquad aaa \qquad\quad -aax \\
\hline
\qquad\qquad\qquad\qquad 2axx-2aax \\
\qquad\qquad\qquad\qquad 2axx-2aax \\
\hline
\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

where we have the same Quotient as before.

The Truth of these Operations is proved as in Division of common Numbers, for if the Work is true, the Quotient being multiplied by the Divisor, the Product will be the given Dividend; thus in the last Example,

$xx+aa-2ax$ is the Quotient.
$-x+a$

$-xxx-aax+2axx$ the Product from multiplying $xx+aa-2ax$, by $-x$.

$axx+aaa-2aax$ the Product from multiplying $xx+aa-2ax$, by a.

$-xxx-3aax+3axx+aaa$ the same with the given Dividend, for though they do not stand in the same Position as in the Example, yet as the Quantities in each Term are alike, and they have the same Co-efficients, and connected by the same Signs, their whole Value, or Amount, must be the same.

The

The Manner of finding the two Roots, or Values, of the unknown Quantity in Quadratic Equations.

61. Now to find the other Value of a, in the *Ambiguous* Quadratic Equation, Queſtion 68.

Take the Work at the Step immediately before you begin to *compleat the Square*, which is at the ninth Step, where the Equation is - - - $ee - se = -p$

Make this Equation equal to *nothing*, by tranſpoſing p - - } $ee - se + p = 0$

Then put it in Numbers, and it is $ee - 45e + 500 = 0$

By the Work we found - - $e = 25$

Make this Equation equal to *nothing*, by tranſpoſing the 25, thus, - - - } $e - 25 = 0$

Then divide $ee - 45e + 500$ by $e - 25$, thus,

$$\begin{array}{l} e - 25)\ ee - 45e + 500\ (e - 20 \\ \quad\quad\ \ \underline{ee - 25e\qquad\qquad} \\ \quad\quad\quad\ \ -20e + 500 \\ \quad\quad\quad\ \ \underline{-20e + 500} \\ \quad\quad\quad\quad\quad\ 0 \end{array}$$

Hence the Quotient is $e - 20$, but as the Dividend is *nothing*, for $ee - 45e + 500 = 0$ as above; and as the Diviſor $e - 25$ is *nothing*, for $e - 25 = 0$ as above, it follows that the Quotient muſt be *nothing*, or equal to *nothing*, that is, $e - 20 = 0$; then tranſpoſing 20, we have $e = 20$, which is the other Value of e, in this Quadratic *Ambiguous* Equation; therefore, I ſay the *youngeſt* Brother was but 20 Years of Age.

And upon this Value of e, if we take the third Step of the Work to the Queſtion, that is, $a = s - e$, we ſhall find $a = 25$, whence the *eldeſt* Brother was 25 Years of Age, and theſe are the true Ages of the two Brothers; for their Ages anſwer the Conditions of the Queſtion, and it is a poſſible Caſe, whereas though the other Numbers anſwered the Conditions of the Queſtion, yet it was impoſſible for the *youngeſt* Brother to be 25, when the *eldeſt* was but 20 Years old.

From the Work of the Queſtion we found -	$e = 25$
But now we have found - -	$e = 20$
The Sum of theſe two Values of e, is - -	45

But obſerving where we put this Quadratic Equation into Numbers, and made it equal to *nothing*, we ſhall find the Co-efficient of the firſt Power of e to be -45, but the Sum of the two Values of e is $+45$, as above, and concerning theſe *Quadratic* Equations, *Algebraiſts* give us this

SCHOLIUM.

62. That in Quadratic Equations *the Sum of both the Roots, or Values, of the unknown Quantity, is equal to the Co-efficient of the loweſt Power of the unknown Quantity, at the Step immediately preceding the compleating the Square, but will have the contrary Sign*; that is, if the Co-efficient of the loweſt Power of the unknown Quantity has the Sign $+$, the Sum of both the Roots will be the ſame as the Co-efficient, but will have the Sign $-$.

And if the Co-efficient of the loweſt Power of the unknown Quantity has the Sign $-$, then the Sum of both the Roots, or Values, will be the ſame as the Co-efficient, but will have the Sign $+$.

Therefore having found any one Root, the other is eaſily found.

63. *To find the other Value of the unknown Quantity in the firſt Form of Quadratic Equations, or where the Co-efficient of the loweſt Power of the unknown Quantity has the Sign* $+$, it is done *by adding the Value of the unknown Quantity found from the Operation, to the Co-efficient of its loweſt Power, and to their Sum prefix the Sign* $-$.

Thus at Queſtion 60, Step 6, the Co-efficient of e, is b, or - - - -	10
To which adding the Value of e, as found by that Operation - - - -	5
The Sum is - - - -	15

And

And prefixing to this 15 the Sign —, and this is the other Value of e, that is, $e = -15$, which is an *imaginary* Value of e, it being abſurd for a *poſitive* Quantity to be equal to a *negative* one.

However, we ſhall find this *imaginary* Value of e, if we proceed by Diviſion according to the Directions at Art. 61.

For the ſixth Step, Queſtion 60, is that which immediately precedes the compleating the Square, where the Equation is - - - $ee + be = m$

Which is in Numbers - - - $ee + 10e = 75$

Tranſpoſe 75, to make the Equation equal to *nothing* - - $ee + 10e - 75 = 0$

By the Work we found - - - $e = 5$

Tranſpoſe 5, to make this Equation equal to *nothing* $e - 5 = 0$

Then dividing $ee + 10e - 75$ by $e - 5$.

$$
\begin{array}{r|l}
e-5) & ee + 10e - 75\ (e + 15 \\
 & ee - 5e \\
\hline
 & \quad 15e - 75 \\
 & \quad 15e - 75 \\
\hline
 & \qquad 0
\end{array}
$$

Hence the Quotient is $e + 15$, but as the Dividend is equal to *nothing*, for $ee + 10e - 75 = 0$, and as the Diviſor $e - 5$ is equal to *nothing*, for $e - 5 = 0$, as above, conſequently the Quotient muſt be equal to *nothing*, that is, $e + 15 = 0$, by tranſpoſing the 15, we have $e = -15$, as before.

For another Example of this Kind take Queſtion 61, where the ſeventh Step is that which immediately precedes the compleating the Square, the Equation being $ee + me = x - b$, which being put in Numbers is - $ee + 6e = 112$

By tranſpoſing 112, to make the Equation equal to *nothing*, we have - $ee + 6e - 112 = 0$

By the Work it was found - - $e = 8$

Tranſpoſing 8, to make the Equation equal to *nothing*, we have - - - - $e - 8 = 0$

And dividing to find the other Root of e, as before,

$$\begin{array}{r} e - 8)\,ee + 6e - 112\,(e + 14 \\ \underline{ee - 8e} \\ 14e - 112 \\ \underline{14e - 112} \\ 0 \end{array}$$

Hence the Quotient is $e + 14$, which for the same Reason as before, it is $e + 14 = 0$, hence $e = -14$, for the other Value of e.

And this Value of e will be found by the Rule Art. 62.

Thus at Question 61, Step 7, the Co-efficient of e is m, or - - - -	6
To which adding the Value of e, found at the Operation - - - - -	8
The Sum is - - -	14

Then by the Rule prefixing the Sign $-$ to 14, we have -14 for the other, or *imaginary* Value of e, the same as before.

But if we add these two Values of e together, we shall find their Sum answer to the *Scholium*, Art. 62.

The first Value of e is - - -	8
The second Value of e is - -	-14
	-6

Hence their Sum is the same with the Co-efficient of e, but has the contrary Sign.

If the Reader has a Mind to prosecute this Speculation, he may try Question 62, Step 9. Question 63, Step 9. Question 64, Step 12, or 13, which are Equations of this first Form, as well as some that follow them.

To find the other Value or Root of the unknown Quantity in the Second Form of Quadratic Equations.

64. The second Form of Quadratic Equations, is when the Co-efficient of the lowest Power of the unknown Quantity has the Sign $-$; in this Case *substract the Co-efficient of the lowest Power, supposing it affirmative, of the unknown Quantity in the given Equation, at the Step immediately preceding the compleating*

he Square, from the Value of the unknown Quantity found by the Work, to the Remainder prefix the Sign —, *and it will be the other Value of the unknown Quantity. Or place down the Co-efficient with its Sign* —, *to which add the Value of the unknown Quantity found by the Work, and to this Sum prefix the Sign* —, *and it will be the other Value, or Root of the unknown Quantity.*

An Equation of this ſecond *Form* is Step 12, Queſtion 65, where we have $aa - za = bd - bmx$.

Here the Co-efficient of a, is $-z$, or - -	-28
And the Value of a found in that Equation is -	$+60$
The Sum is 32, but to it prefix the Sign —, and it is -32, the other Value of a, which is *imaginary*, as it has the Sign —.	-32

And if we proceed by Diviſion according to the Directions at Art. 61. we ſhall find this *imaginary* Value of a.

Thus if we take the twelfth Step of Queſtion 65, which immediately precedes compleating the Square, we have this Equation - - $aa - za = bd - bmx$

Which being put in Numbers is - $aa - 28a = 1920$

Tranſpoſing 1920 to make the Equation equal to *nothing* - $aa - 28a - 1920 = 0$

By the Work it was found - - $a = 60$

Tranſpoſe 60 to make the Equation equal to *nothing* - - - $a - 60 = 0$

And dividing to find the other Root of a, as before,

$$\begin{array}{r|l} a - 60) & aa - 28a - 1920\ (a + 32 \\ & \underline{aa - 60a} \\ & \quad\ 32a - 1920 \\ & \quad\ \underline{32a - 1920} \\ & \qquad\quad 0 \end{array}$$

Hence the Quotient is $a + 32$, which becauſe the Dividend and Diviſor are each equal to *nothing*, conſequently the Quotient muſt be equal to *nothing*, hence - $a + 32 = 0$

By tranſpoſing 32, we have - $a = -32$, the ſame *imaginary* Value of a, as before.

And

And if we add these two Values of a together, we shall find their Sum agree with the *Scholium*, Art. 62.

The Value of a found by the Operation, Question 65, is	60
The Value of a now found is - -	-32
Their Sum is 28, or $+28$, the same Number as the Co-efficient of a, but with a contrary Sign -	28

Another Equation of this second *Form* is Question 67, Step 13, where the Equation is $aa - pa = \frac{zd - rm}{zx}$

Which being put into Numbers is - $aa - 1.8a = 82$

Transposing 82, to make the Equation equal to *nothing* - - $aa - 1.8a - 82 = 0$

By the Work it was found - - $a = 10$

Transposing 10 to make the Equation equal to *nothing* - - - - $a - 10 = 0$

And dividing to find the other Value of a, as before,

$$
\begin{array}{r}
a - 10)\ aa - 1.8a - 82\ (a + 8.2 \\
aa - 10a \qquad\quad \\
\hline
8.2a - 82 \\
8.2a - 82 \\
\hline
0
\end{array}
$$

Hence the Quotient is $a + 8.2$ which must be equal to *nothing*, for the Dividend and Divisor are each equal to *nothing:* but if $a + 8.2 = 0$,

By transposing 8.2 we have $a = -8.2$ which is the other Value of a, and it is *imaginary*, because it has the Sign $-$.

The same *imaginary* Value of a may be found by Art. 64, thus,

The Co-efficient of a, is - -	-1.8
The Value of a found by the Question, is -	10.
The Sum is - - -	8.2

Now to this 8.2 prefix the Sign $-$, and we have -8.2 for the *imaginary* Value of a, the same as before.

And

And if thefe two Values of a are added together, their Sum will agree with the *Scholium*, Art. 62.

The firft Value of a, is - -	10.
The fecond Value of a, is - -	— 8.2
Sum - - -	- 1.8

But the Co-efficient of a, is — 1.8

65. But in the *ambiguous*, or third *Form* of Quadratic Equations,

If the Value of the unknown Quantity found by the Operation, is fubftracted from the Co-efficient of its loweft Power, at the Step immediately before the Square is compleated, the Co-efficient being fuppofed affirmative, the Remainder is its other Value.

At Queftion 68, Step 9, the Co-efficient of e is s, or	45
The Value of e, found by the Operation, is -	25
The Remainder is the other Value of e - -	20

And it is this fecond Value of e that is the true Anfwer to the Queftion, as was obferved Page 187; and here the Learner may again obferve, that both the Values in this Cafe are *affirmative*, which makes this be called the *ambiguous* Cafe, but in the other two preceding Cafes, or in the four former Examples, the other Value of the unknown Quantity was *negative*, which is only an *imaginary* Value, it being impoffible for an *affirmative*, or *pofitive* Quantity, which the Queftion requires, to be a *negative*, or equal to a *negative* Quantity.

But we may find the other Value of e, in this *ambiguous* Cafe, by Divifion, as in the former Inftances, thus,

The Equation, Queftion 68, Step 9, immediately before the Square was compleated, is - - - $\}$ $ee - se = -p$

Which being put in Numbers, is - $ee - 45e = -500$

Tranfpofing 500, to make the Equation equal to *nothing* $\}$ $ee - 45e + 500 = 0$

By the Work it was found - - $e = 25$

Tranfpofing 25, to make the Equation equal to *nothing* - - - $\}$ $e - 25 = 0$

And dividing to find the other Value, or Root of e, as before,

$$
\begin{array}{r}
e - 25)\ ee - 45e + 500\ (e - 20 \\
ee - 25e \\
\hline
-20e + 500 \\
-20e + 500 \\
\hline
0
\end{array}
$$

Hence the Quotient is $e - 20$, which muſt be equal to *nothing*, for the Reaſon in the former Caſes, but if $e - 20 = 0$

Tranſpoſing 20, we have - - $e = 20$ the other Value of e, the ſame as before.

And in this *ambiguous* Caſe, if we add the two Values of e together, we ſhall find them agree with what is ſaid at the *Scholium*, Art. 62.

The firſt Value of e, is - -	25
The ſecond Value of e, is - - - -	20
	45

But the Co-efficient of e, is -45.

The Manner of expreſſing the two Roots of an ambiguous Quadratic Equation explained.

66. Now to explain the uſual Manner in which *Algebraiſts* expreſs the Value of the unknown Quantity, in the *ambiguous* Quadratic Equation; let us reſume the Solution of Queſtion 68, at the eighth Step, where there is this

Equation	1	$ee = se - p$
$1 - se$	2	$ee - se = -p$
$2\ c\ \square$	3	$ee - se + \frac{ss}{4} = \frac{ss}{4} - p$
$3\ w\ 2$	4	$e - \frac{s}{2} = \sqrt{\frac{ss}{4} - p}$
$4 + \frac{s}{2}$	5	$e = \frac{s}{2} \pm \sqrt{\frac{ss}{4} - p}$

That

That is, *prefix both the Signs* $+$ *and* $-$, *to the Quantity under the radical Sign, for that being added to* $\frac{s}{2}$, *or the rational Quantities on that Side of the Equation, gives* one *of the Values of* e, *but if it is subtracted from* $\frac{s}{2}$, *or the rational Quantities on that Side of the Equation, then it gives the* other *Value of* e, *thus,*

$$\begin{array}{r} s = 45 \\ s = 45 \\ \hline 225 \\ 180 \\ \hline 4)\ 2025 = ss \\ \hline 506.25 = \frac{ss}{4} \\ -p = 500. \\ \hline 6.25 = \frac{ss}{4} - p\ (2.5 = \sqrt{\frac{ss}{4} - p} \\ 4 \\ \hline 45)\ 225 \\ 225 \\ \hline 0 \end{array}$$

Then to find the two Values or Roots of *e*.

$$\begin{array}{r} \frac{s}{2} = 22.5 \\ +\sqrt{\frac{ss}{4} - p} = 2.5 \\ \hline 25. \end{array} = \text{one of the Values of } e.$$

$$\begin{array}{r} \frac{s}{2} = 22.5 \\ -\sqrt{\frac{ss}{4} - p} = -2.5 \\ \hline 20. \end{array}$$

the other Value of *e*, which two Values of *e* are the same as was found at Art. 61.

And this is the common Method in which *Algebraists* set down, or express the Value of the unknown Quantity, in the *ambiguous* Quadratic Equation.

The Reason of Quadratic Equations having two different Values of the same unknown Quantity, is because the same Quadratic Equation can be formed from two different Suppositions, or Values of the unknown Quantity, or supposing the same unknown Quantity to be equal to two different Numbers.

For let us resume the Equation $ee - se = -p$, or $ee - 45e = -500$, in this *ambiguous* Equation we found the first Value of e to be 25, by making e equal to 25, we have

	1	$e = 25$
1 ⓖ 2	2	$ee = 625$
Multiplying the first Equation by -45, the Co-efficient of e, in the given Equation -	3	$-45e = -1125$
$2 + 3$	4	$ee - 45e = -500$, the same with the given Quadratic Equation.

And if we take the other Value of e, *viz.* 20, we can form the given Equation, for

Let	1	$e = 20$
1 ⓖ 2	2	$ee = 400$
1×-45 the Co-efficient of e, in the given Equation -	3	$-45e = -900$
$2 + 3$	4	$ee - 45e = -500$, the same with the given Quadratic Equation.

Likewise if we take the first *Form* of Quadratic Equations, *viz.* $ee + be = m$, or $ee + 10e = 75$, see Question 60, Step 6. Now the two Values of e in this Equation we found to be 5, and -15, and from either of these Values of e, we can form the given Quadratic Equation,

Suppose	1	$e = 5$
1 ⓖ 2	2	$ee = 25$

1 ×

$1 \times \overline{10}$ the Co-efficient of e, in the given Equation - -	3	$10e = 50$
$2 + 3$	4	$ee + 10e = 75$, the same with the given Quadratic Equation.

Again, suppose	1	$e = -15$
1 & 2	2	$ee = 225$, for $-15 \times -15 = +225$, the Signs being alike.
$1 \times \overline{10}$ as above	3	$10e = -150$
$2 + 3$	4	$ee + 10e = 75$, the same Equation as before.

And if we take the second *Form* of Quadratic Equations, *viz.* $aa - za = bd - bmx$, or $aa - 28a = 1920$, see Question 65, Step 12. The two Values of a in this Equation we found to be 60, and -32, from either of which we can form the given Equation, for

Suppose	1	$a = 60$
1 & 2	2	$aa = 3600$
$1 \times -\overline{28}$ the Co-efficient of a, in the given Equation -	3	$-28a = -1680$
$2 + 3$	4	$aa - 28a = 1920$, the same with the given Equation.

Again, if	1	$a = -32$
1 & 2	2	$aa = 1024$, for $-32 \times -32 = +1024$.
$1 \times -\overline{28}$ the Co-efficient of a, as above	3	$-28a = 896$, for $-28 \times -32 = +896$.
$2 + 3$	4	$aa - 28a = 1920$, the same with the given Equation.

From this the Learner may observe, *that making the unknown Quantity equal to either of its Values, and raising this Equation to the Square, and adding it to the former Equation, after it has been multiplied by the Co-efficient of the lowest Power of the unknown Quantity in the Quadratic Equation, this Sum will be the given Quadratic Equation.*

Question

Queſtion 69. *Two Men,* A *and* B, *diſcourſing of their Shillings,* A, *who had the greateſt Number, ſaid, if my Number of Shillings is divided by your's, and this Quotient is added to your Number of Shillings, the Sum will be* 15:

But if the Sum of both our Shillings is multiplied by 4, *and this Product divided by* 10, *the Quotient will be* 22. *How many Shillings had each Perſon?*

Let $a =$ the Number of Shillings A had, or the greateſt Number, $e =$ the Number of Shillings B had, or leſſer Number, $s = 15$, $m = 4$, $n = 10$, $d = 22$.

Then	1	$\frac{a}{e} + e = s$	By the Queſtion.
And	2	$\frac{ma + me}{n} = d$	
$1 \times e$	3	$a + ee = se$	
$3 - ee$	4	$a = se - ee$	
$2 \times n$	5	$ma + me = dn$	
$5 - me$	6	$ma = dn - me$	
$6 \div m$	7	$a = \frac{dn - me}{m}$	
$4 \cdot 7$	8	$\frac{dn - me}{m} = se - ee$	
$8 \times m$	9	$dn - me = mse - mee$	
$9 + mee$	10	$mee + dn - me = mse$	
$10 - dn$	11	$mee - me = mse - dn$	
$11 - mse$	12	$mee - me - mse = -dn$	

Here the Equation appears to be *Quadratic*, and of the *ambiguous* Kind; becauſe dn, the known Side of the Equation, has the *negative* Sign. Then by Art. 58, dividing by m, the Coefficient of ee,

$12 \div m$	13	$ee - e - se = -\frac{dn}{m}$.

For m being in every Term on one Side of the Equation, dividing that Part of the Equation by m, is only to caſt away m, out of every Term of that Side of the Equation, and to divide the other Side

Side of the Equation is only to place m as a Denominator to it. The Equation being now prepared for *compleating the Square*, and the firſt Power of e being in two Terms, *viz.* $-e-se$, whoſe Co-efficients are -1, and $-s$,

Therefore		Subſtitute $-z=-1-s$, then by
Subſtitution	14	$ee-ze=-\frac{dn}{m}$
$14\ c\ \square$	15	$ee-ze+\frac{zz}{4}=\frac{zz}{4}-\frac{dn}{m}$
$15\ ш\ 2$	16	$e-\frac{z}{2}=\sqrt{\frac{zz}{4}-\frac{dn}{m}}$
$16+\frac{z}{2}$	17	$e=\frac{z}{2}\pm\sqrt{\frac{zz}{4}-\frac{dn}{m}}=8\pm3$,
		(that is, e is either 5, or 11.

And if e is 5, we ſhall find $a=50$, by the fourth Step, which two Numbers of Shillings anſwer the Conditions of the Queſtion; or, if we ſuppoſe $e=11$, then by the fourth Step we ſhall find $a=44$, which two Numbers will likewiſe anſwer the Conditions of the Queſtion: But ſometimes one of the Numbers, or Roots, of theſe ambiguous Equations, will not anſwer all the Conditions of the Queſtion, as at Queſtion 74, and then the other Root muſt be found.

Queſtion 70. *Two Merchants,* A *and* B, *had gained in Trade, but* A, *who gained the moſt, found, that if the Square of the Pounds he gained was multiplied by* 2, *and the Product added to* 8 *times the Pounds* B *gained, if this Sum was divided by* 4, *the Quotient was* 816 *Pounds:*

But if 3 *times the Pounds* A *gained, was added to* 10 *times the Pounds* B *gained, and this Sum divided by* 40, *the Quotient was* 5 *Pounds. How many Pounds had each Man gained?*

Put $a=$ the Pounds gained by A, $e=$ the Pounds gained by B, $x=2$, $m=8$, $p=4$, $d=816$, $b=3$, $z=10$, $r=40$, $n=5$.

	1	$\frac{xaa+me}{p}=d$	By the Question.
	2	$\frac{ba+ze}{r}=n$	
$1\times p$	3	$xaa+me=pd$	
$3-xaa$	4	$me=pd-xaa$	
$4\div m$	5	$e=\frac{pd-xaa}{m}$	
$2\times r$	6	$ba+ze=rn$	
$6-ba$	7	$ze=rn-ba$	
$6\div z$	8	$e=\frac{rn-ba}{z}$	
$5\cdot 8$	9	$\frac{rn-ba}{z}=\frac{pd-xaa}{m}$	
$9\times z$	10	$rn-ba=\frac{zpd-zxaa}{m}$	
$10\times m$	11	$mrn-mba=zpd-zxaa$	

Tranſpoſe $zxaa$, that the higheſt Power of the unknown Quantity may have the Sign $+$.

$11+zxaa$	12	$zxaa+mrn-mba=zpd$
$12-mrn$	13	$zxaa-mba=zpd-mrn$

The Equation now appears to be Quadratic, but to know if it is *ambiguous*, find which Quantity is greateſt zpd, or mrn, but zpd is 32640, and mrn is only 1600, hence $zpd-mrn=32640-1600=31040$, which being an *affirmative* Number, the Equation is not ambiguous, by Art. 59. But becauſe the Square of the unknown Quantity has a Co-efficient, therefore, by Art. 58,

$13\div zx$	14	$aa-\frac{mba}{zx}=\frac{zpd-mrn}{zx}$
		Subſtitute $-s=-\frac{mb}{zx}$ then by
Subſtitution	15	$aa-sa=\frac{zpd-mrn}{zx}$
$15\ c\ \square$	16	$aa-sa+\frac{ss}{4}=\frac{zpd-mrn}{zz}+\frac{ss}{4}$

2 16 w

$16 \text{ w} 2$	17	$a-\frac{s}{2}=\sqrt{\frac{zpd-mrn}{zx}+\frac{ss}{4}}$
$17+\frac{s}{2}$	18	$a=\sqrt{\frac{zpd-mrn}{zx}+\frac{ss}{4}}:+\frac{s}{2}$ (= 40, the Pounds gained by A.
By the eighth Step	19	$e=\frac{rn-ba}{z}=8$, the Pounds gained (by B.

Queſtion 71. *A Father, by his Will, left his two Sons,* A *and* B, *ſuch a Portion, whereof* A *had the greateſt Fortune; that if the Square of the Number of Pounds he was to have, be multiplied by* 2, *and to this Product there is added the Number of Pounds* B *was to have multiplied by* 35, *the Sum was* 6400 *Pounds:*

But if the Number of Pounds A *was to have, be multiplied by* 20, *and this Product added to the Number of Pounds* B *was to have multiplied by* 15, *the Sum was* 1600 *Pounds. To find the Fortune of each?*

Let $a=$ the Fortune of A, $e=$ the Fortune of B, $x=2$, $m=35$, $d=6400$, $b=20$, $z=15$, $r=1600$.

	1	$xaa+me=d$ } By the Queſtion.
	2	$ba+ze=r$
$1-xaa$	3	$me=d-xaa$
$3\div m$	4	$e=\frac{d-xaa}{m}$
$2-ba$	5	$ze=r-ba$
$5\div z$	6	$e=\frac{r-ba}{z}$
$4 . 6$	7	$\frac{r-ba}{z}=\frac{d-xaa}{m}$
$7\times z$	8	$r-ba=\frac{zd-zxaa}{m}$
$8\times m$	9	$mr-mba=zd-zxaa$

Tranſpoſe $zxaa$, that the higheſt Power of a may be *affirmative.*

$9+zxaa$	10	$zxaa+mr-mba=zd$
$10-mr$	11	$zxaa-mba=zd-mr$

 The

The Equation now appears to be *Quadratic*, and to know if it is *ambiguous*, find what zd and mr are in Numbers. But $zd - mr = 96000 - 56000 = 40000$, a *positive* Quantity, whence the Equation is not *ambiguous* by Art. 59. And because the Square of the unknown Quantity has a Co-efficient, therefore by Art. 58.

$11 \div zx$	12	$aa - \frac{mba}{zx} = \frac{zd - mr}{zx}$
		Substitute $-s = -\frac{mb}{zx}$ then by
Substitution	13	$aa - sa = \frac{zd - mr}{zx}$
$13\ c\ \square$	14	$aa - sa + \frac{ss}{4} = \frac{zd - mr}{zx} + \frac{ss}{4}$
$14\ w\ 2$	15	$a - \frac{s}{2} = \sqrt{\frac{zd - mr}{zx} + \frac{ss}{4}}$
$15 + \frac{s}{2}$	16	$a = \sqrt{\frac{zd - mr}{zx} + \frac{ss}{4}} : + \frac{s}{2}$

$= 49.9999$, &c. because of the Imperfection of the Decimal Fraction; the true Number being 50, from which by

The sixth Step	17	$e = \frac{r - ba}{z} = 40.$

Question 72. *What are those two Numbers, the Quotient of the greater divided by* 5, *and added to the lesser, the Sum may be* 12:

But the Product of the two Numbers divided by 4, *the Quotient is* 40?

Put $a =$ the greater Number, $e =$ the lesser Number, $m = 5$, $p = 12$, $d = 4$, $x = 40$.

	1	$\frac{a}{m} + e = p$ } By the Question.
	2	$\frac{ae}{d} = x$ }
$1 \times m$	3	$a + me = mp$
$3 - me$	4	$a = mp - me$
$2 \times d$	5	$ae = dx$

$5 \div e$

$5 \div e$	6	$a = \frac{dx}{e}$
$4 . 6$	7	$\frac{dx}{e} = mp - me$
$7 \times e$	8	$dx = mpe - mee$
$8 + mee$	9	$mee + dx = mpe$
$9 - dx$	10	$mee = mpe - dx$
$10 - mpe$	11	$mee - mpe = -dx$

Here the Equation not only appears *Quadratic*, but *Ambiguous*, for dx the known Side of the Equation is *negative*. Now by Art. 58.

$11 \div m$	12	$ee - pe = -\frac{dx}{m}$
12 c $\square$	13	$ee - pe + \frac{pp}{4} = \frac{pp}{4} - \frac{dx}{m}$
13 ꟺ 2	14	$e - \frac{p}{2} = \sqrt{\frac{pp}{4} - \frac{dx}{m}}$
$14 + \frac{p}{2}$	15	$e = \frac{p}{2} \pm \sqrt{\frac{pp}{4} - \frac{dx}{m}} = 6 \pm 2$, that is, e is either 8, or 4: But if $e = 8$, then by
The fixth Step	16	$a = \frac{dx}{e} = 20$. Or if $e = 4$, then

$a = 40$, either of which anfwers the Queftion.

Queftion 73. *Two young Gentlemen having been at the Gaming-Tables, and being afked by their Friend what they loft, which being afhamed to own,* A *faid, if the Number of Pounds I loft is divided by* 4, *and this added to the Number of Pounds* B *loft divided by* 2, *the Sum is* 9 *Pounds:*

But if the Product of the Number of Pounds we both loft is divided by 10, *and extracting the fquare Root of this Quotient, it will be* 4. *How much did each Perfon lofe?*

Let $a =$ the Number of Pounds loft by A, $e =$ the Number of Pounds loft by B, $b = 4$, $d = 2$, $m = 9$, $p = 10$: as the Number 4 is in the firft Part of the Queftion, and it being again repeated, there is no Occafion for any new Letter.

	1	$\frac{a}{b}+\frac{e}{d}=m$	By the Question.
	2	$\sqrt{\frac{ae}{p}}=b$	
$1-\frac{e}{d}$	3	$\frac{a}{b}=m-\frac{e}{d}$	
$3\times b$	4	$a=bm-\frac{be}{d}$	
$2 \oplus 2$	5	$\frac{ae}{p}=bb$	
$5\times p$	6	$ae=pbb$	
$6\div e$	7	$a=\frac{pbb}{e}$	
$4\cdot 7$	8	$\frac{pbb}{e}=bm-\frac{be}{d}$	
$8\times e$	9	$pbb=bme-\frac{bee}{d}$	
$9\times d$	10	$dpbb=dbme-bee$	
		Dividing by b, by Art. 53.	
$10\div b$	11	$dpb=dme-ee$	
		To have the highest Power of e *affirmative*, transpose ee.	
$11+ee$	12	$ee+dpb=dme$	
$12-dpb$	13	$ee=dme-dpb$	
$13-dme$	14	$ee-dme=-dpb$	
		Here the Equation appears *quadratic*, and *ambiguous*.	
$14\ c\ \square$	15	$ee-dme+\frac{ddmm}{4}=\frac{ddmm}{4}-dpb$	
$15\ ɯ\ 2$	16	$e-\frac{dm}{2}=\sqrt{\frac{ddmm}{4}-dpb}$	
$16+\frac{dm}{2}$	17	$e=\frac{dm}{2}\pm\sqrt{\frac{ddmm}{4}-dpb}=9\pm 1,$	

that is, e is either 8, or 10, whence by the fourth, or seventh Steps, we shall find $a=20$, or 16.

Question 74. *In the right-angled Triangle* ABC, *there is given the Hypothenuse* AC = 10, *and the Sum of the Base* AB *and Perpendicular* BC = 14. *To find the Base* AB *and Perpendicular* BC? See Figure, Page 206.

Let

Let $AC = b = 10$, $AB + BC = d = 14$, $AB = a$; and becaufe $AB + BC = d$, therefore from this fubftracting AB, or a, we have $BC = d - a$.

Having expreffed all the Sides of the Figure in Symbols, and there being but one unknown Quantity, we are only to raife one Equation from the Property of the Figure; and the Triangle ABC being right-angled, we have by 47 *e* 1 the Square of the Hypothenufe AC, or bb, equal to the Square of the Bafe AB, or aa, added to the Square of the Perpendicular BC, or $dd - 2da + aa$, that is,

In Symbols	1	$bb = dd - 2da + 2aa$
$1 - dd$	2	$2aa - 2da = bb - dd$

Here the Equation appears *quadratic*, and becaufe dd is greater than bb, it is likewife *ambiguous*, for $bb - dd = 100 - 196 = -96$ a *negative* Quantity; but as the Square of the unknown Quantity has a Co-efficient, therefore divide by it by Art. 58.

$2 \div \bar{2}$	3	$aa - da = \dfrac{bb - dd}{2}$
$3\ c\ \square$	4	$aa - da + \dfrac{dd}{4} = \dfrac{dd}{4} + \dfrac{bb - dd}{2}$
4 ш 2	5	$a - \dfrac{d}{2} = \sqrt{\dfrac{dd}{4} + \dfrac{bb - dd}{2}}$
$5 + \dfrac{d}{2}$	6	$a = \dfrac{d}{2} \pm \sqrt{\dfrac{dd}{4} + \dfrac{bb - dd}{2}} = 7 \pm 1$

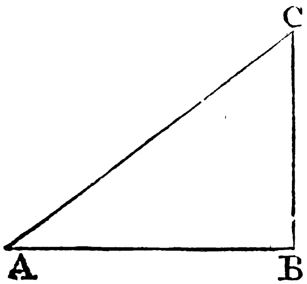
from whence the Bafe AB may be either 8, or 6; fuppofing the Bafe 8, then becaufe by the Queftion, the Sum of the Bafe and Perpendicular is 14, the Perpendicular BC will be 6; but if we fuppofe the Bafe to be 6, then from the fame Reafoning the Perpendicular BC will be 8.

And the Queftion not limiting which is longeft, either the Bafe AB, or Perpendicular BC, we may take either 6, or 8, for the Length of the Bafe AB, for either will anfwer the Conditions of the Queftion.

But if the Queftion had faid that the Bafe AB, is longer than the Perpendicular BC, then we muft take $a = \dfrac{d}{2} + \sqrt{\dfrac{dd}{4} + \dfrac{bb - dd}{2}} = 8$, by which we fhall find the Perpen-

dicular

dicular BC $= 6$; for if we take $a = \frac{d}{2} - \sqrt{\frac{dd}{4} + \frac{bb - dd}{2}} = 6$, then we shall find the Perpendicular BC $= 8$, which cannot be, because the Question is supposed to determine the Base AB, to be longer than the Perpendicular BC.



Question 75. *In the right-angled Triangle* ABC, *given the Hypothenuse* AC $= 10$, *the Perpendicular* BC, *being shorter than the Base* AB, *by substracting the Perpendicular* BC *from the Base* AB, *and multiplying the Difference by* 20, *and dividing this Product by* 8, *the Quotient is* 5. *What is the Length of the Base* AB, *and Perpendicular* BC?

Let AC $= b = 10$, AB $= a$, BC $= e$, $d = 20$, $m = 8$, $z = 5$.

	1	$aa + ee = bb$ by the Property of the Figure, as in the last Question.
	2	$\frac{da - de}{m} = z$ by the Question.
$1 - ee$	3	$aa = bb - ee$
$3\ \omega\ 2$	4	$a = \sqrt{bb - ee}$
$2 \times m$	5	$da - de = zm$
$5 + de$	6	$da = zm + de$
$6 \div d$	7	$a = \frac{zm + de}{d}$
$4 \cdot 7$	8	$\frac{zm + de}{d} = \sqrt{bb - ee}$

Squaring both Sides of the Equation, because the unknown Quantity is under the radical Sign.

$8 \odot 2$	9	$\frac{zzmm + 2zmde + ddee}{dd} = bb - ee$
$9 \times dd$	10	$zzmm + 2zmde + ddee = ddbb - ddee$
$10 + ddee$	11	$zzmm + 2zmde + 2ddee = bbdd$
$11 - zzmm$	12	$2ddee + 2zmde = bbdd - zzmm$
2		

Dividing

		Dividing by the Co-efficient of *ee* by Art. 58.
$12 \div 2dd$	13	$ee + \frac{2zmde}{2dd} = \frac{bbdd - zzmm}{2dd}$
That is	14	$ee + \frac{zme}{d} = \frac{bbdd - zzmm}{2dd}$ for $\frac{2zmde}{2dd} = \frac{zme}{d}$, rejecting $2d$, as in Division.

Hence the Equation is *quadratic*, but it cannot be *ambiguous*, becauſe both the Quantities $ee + \frac{zme}{d}$ having the Sign $+$, the whole Side of the Equation muſt be *affirmative*, and conſequently the other Side of the Equation muſt be alſo *affirmative*, otherwiſe an *affirmative* Quantity would be equal to a *negative* Quantity, which is abſurd. Now,

		Subſtitute $x = \frac{zm}{d} = 2$, by Art. 57. then by
Subſtitution	15	$ee + xe = \frac{bbdd - zzmm}{2dd}$
$15\ c\ \square$	16	$ee + xe + \frac{xx}{4} = \frac{bbdd - zzmm}{2dd} + \frac{xx}{4}$
16 ɯ 2	17	$e + \frac{x}{2} = \sqrt{\frac{bbdd - zzmm}{2dd} + \frac{xx}{4}}$
$17 - \frac{x}{2}$	18	$e = \sqrt{\frac{bbdd - zzmm}{2dd} + \frac{xx}{4}} : - \frac{x}{2} = 6 = BC.$
Then by Step 7th	19	$a = \frac{zm + de}{d} = 8 = AB.$

The ſame Queſtion done otherwiſe.

Let $AC = b = 10$, $AB = a$, then by 47 *e* 1, $BC = \sqrt{bb - aa}$; ſuppoſe $d = 20$, $m = 8$, $z = 5$, as before.

Now

Now all the Sides of the Triangle being expreſſed in Symbols, and there being only one unknown Quantity, there is but one Equation required, which may be raiſed from the Conditions of the Queſtion, and theſe I ſhall particularly expreſs to prevent any Difficulty to the Learner.

Now | 1 | a, is the Baſe AB, which is longer than the Perpendicular BC, or $\sqrt{bb - aa}$, therefore connecting $\sqrt{bb - aa}$ to a by the Sign —,

We have | 2 | $a - \sqrt{bb - aa}$ for the Difference between the Baſe and Perpendicular, which is to be multiplied by 20, or d, then

We have | 3 | $da - d\sqrt{bb - aa}$

But this Product is to be divided by 8, or m, then

We have	4	$\frac{da - d\sqrt{bb - aa}}{m}$, and this Quotient (is to be equal to 5, or z.
Whence	5	$\frac{da - d\sqrt{bb - aa}}{m} = z$, by the (Queſtion.

Becauſe the unknown Quantity is divided by m, therefore by Art. 47.

$5 \times m$	6	$da - d\sqrt{bb - aa} = zm$

Becauſe the unknown Quantity is multiplied by d, therefore by Art. 48.

$6 \div d$	7	$a - \sqrt{bb - aa} = \frac{zm}{d}$

Now tranſpoſe the Surd, becauſe it has the Sign —, the higheſt Power of the unknown Quantity being Part of it.

$7 + \sqrt{bb - aa}$	8	$a = \frac{zm}{d} + \sqrt{bb - aa}$
$8 - \frac{zm}{d}$	9	$\sqrt{bb - aa} = a - \frac{zm}{d}$

There being no Quantities on the ſame Side of the Equation with the Surd, raiſe both Sides of the Equation to the ſecond Power to take away the radical Sign.

$9 \circledS 2$	10	$bb - aa = aa - \frac{2zma}{d} + \frac{zzmm}{dd}$
$10 + aa$	11	$2aa - \frac{2zma}{d} + \frac{zzmm}{dd} = bb$
$11 - \frac{zzmm}{dd}$	12	$2aa - \frac{2zma}{d} = bb - \frac{zzmm}{dd}$
$12 \div 2$	13	$aa - \frac{zma}{d} = \frac{bb}{2} - \frac{zzmm}{2dd}$

Here the Equation is *quadratic*, but becaufe $\frac{bb}{2} - \frac{zzmm}{2dd}$ $= 50 - 2 = 48$, a *pofitive* Quantity, it is not *ambiguous.* Now by Art. 57, fubftitute $-x = -\frac{zm}{d} = -2$.

Then	14	$aa - xa = \frac{bb}{2} - \frac{zzmm}{2dd}$
$14\, c\, \square$	15	$aa - xa + \frac{xx}{4} = \frac{xx}{4} + \frac{bb}{2} - \frac{zzmm}{2dd}$
$15\ \omega\ 2$	16	$a - \frac{x}{2} = \sqrt{\frac{xx}{4} + \frac{bb}{2} - \frac{zzmm}{2dd}}$
$16 + \frac{x}{2}$	17	$a = \frac{x}{2} + \sqrt{\frac{xx}{4} + \frac{bb}{2} - \frac{zzmm}{2dd}}$ (= 8 = the Bafe AB, as before.

Hence in the right-angled Triangle ABC, becaufe we have given the Hypothenufe AC, which is 10, and having now found the Bafe AB to be 8, therefore the Perpendicular BC $= \sqrt{100 - 64} = 6$.

Queftion 76. *Two Merchants,* A *and* B, *becoming Bankrupts, owe fuch Sums of Money, that if from the Number of Pounds* A *owes, we fubftract the Square of the Number of Pounds* B *owes, there remains* 1900 *Pounds :*

But if the Square of the Number of Pounds B *owes, is multiplied by the Number of Pounds* A *owes, the Product is* 81000000 *Pounds. To find the Debt of each Merchant ?*

 Let

Let $a =$ the Money owed by A, $e =$ the Money owed by B, $b = 1900$, $d = 81000000$.

	1	$a - ee = b$ } By the Queſtion.
	2	$aee = d$
$1 + ee$	3	$a = b + ee$
$2 \div ee$	4	$a = \frac{d}{ee}$
$3 \cdot 4$	5	$ee + b = \frac{d}{ee}$
$5 \times ee$	6	$eeee + bee = d$

Perhaps this Equation may appear new to the young *Analyſt*, but by turning to Art. 56. he will find it to be a *Quadratic* Equation, for the unknown Quantity is only in two Terms, and in one of them its Power or Height is *double* its Power or Height in the other, for it is $eeee$ and ee, therefore take b the Co-efficient of ee, the loweſt Power of e in the preſent Caſe; divide it by 2, ſquare the Quotient, and add it to both Sides of the Equation, as before, thus,

$6 c \square$	7	$eeee + bee + \frac{bb}{4} = d + \frac{bb}{4}$
		Extracting the ſquare Root as uſual,
$7 \omega 2$	8	$ee + \frac{b}{2} = \sqrt{d + \frac{bb}{4}}$
		Tranſpoſing $\frac{b}{2}$ becauſe it is a known Quantity,
$8 - \frac{b}{2}$	9	$ee = \sqrt{d + \frac{bb}{4}} : - \frac{b}{2}$
		Now extracting the ſquare Root to depreſs e to the firſt Power.
$9 \omega 2$	10	$e = \sqrt{\sqrt{d + \frac{bb}{4}} : - \frac{b}{2}} = 90$ Pounds, (the Money B owed.

To extract the Square Root of the Quantity $\sqrt{d + \frac{bb}{4}} - \frac{b}{2}$, is only to place again the radical Sign before the ſame

Quantity,

Quantity, drawing it over the radical Sign already there, and the other Quantities without that Sign, if there are any; for though theſe were not included in the firſt Root, yet as they were afterwards tranſpoſed to that Side of the Equation, and the Root is again required to be taken, they will now be included under the radical Sign of this ſecond Extraction.

Becauſe of the two radical Signs, I ſhall ſet down the Numerical Work, to make the Operation the plainer.

$$d = 81000000$$
$$\frac{bb}{4} = 902500$$
$$81902500 = d + \frac{bb}{4} \quad (9050 = \sqrt{d + \frac{bb}{4}}$$
$$81$$
$$1805)\,9025$$
$$9025$$
$$0$$

$$-950 = -\frac{b}{2}$$
$$8100 = \sqrt{d + \frac{bb}{4}} : -\frac{b}{2}$$

and the ſquare Root of 8100 is $90 = \sqrt{\sqrt{d + \frac{bb}{4}} : -\frac{b}{2}}$.

Then by the third Step $a = b + ee$ $= 10000$ Pounds, the Money owed by A.

The ſame Queſtion anſwered by exterminating the unknown Quantity e.

	1	$a - ee = b$	By the Queſtion, as before.
	2	$aee = d$	
$1 - ee$	3	$a = b + ee$	
$3 - b$	4	$ee = a - b$	
$2 \div a$	5	$ee = \frac{d}{a}$	

Make the fourth and fifth Equations equal to one another, for each is equal to ee.

$4 \cdot 5$	6	$a - b = \frac{d}{a}$
$6 \times a$	7	$aa - ba = d$

$7\ c\ \square$	8	$aa - ba + \frac{bb}{4} = d + \frac{bb}{4}$
$8\ \omega\ 2$	9	$a - \frac{b}{2} = \sqrt{d + \frac{bb}{4}}$
$9 + \frac{b}{2}$	10	$a = \sqrt{d + \frac{bb}{4}} : + \frac{b}{2} = 10000$

(Pounds, as before.

And by the fourth Step $e = \sqrt{a - b}$, or by the fifth Step, $e = \sqrt{\frac{d}{a}} = 90$ Pounds, as before.

From hence the Learner may obſerve, there are different Methods of anſwering the ſame Queſtion, and that ſome are more elegant than others, as they give the Anſwer in more ſimple or leſs complicated Terms: And in this Part of the Science he is to exerciſe himſelf according to his own Prudence and Judgment, and ſome Meaſure in Proportion as he underſtands and conceives the general and univerſal Methods by which Queſtions are anſwered; it being only my Deſign to illuſtrate theſe by pertinent Examples, with ſuch Solutions as ariſe in an obvious Manner from the Directions, that the Learner may acquire ſome general Idea of the Nature and Excellency of Algebra.

Queſtion 75. *Two Running Footmen,* A *and* B, *meeting on the Road, found, if the Number of Miles* A *had run was multiplied by* 5, *and ſubſtracting from this Product the Square of the Miles run by* B, *there remained* 100:

But if the Square of the Miles run by B, *was multiplied by the Number of Miles run by* A, *and the Product multiplied by* 2, *this Product was* 80000. *How many Miles had each Perſon run?*

Let $a =$ the Number of Miles run by A, $e =$ the Number of Miles run by B, $m = 5$, $x = 100$, $d = 2$, $b = 80000$.

	1	$ma - ee = x$	By the Queſtion.
	2	$daee = b$	
$1 + ee$	3	$ma = x + ee$	
$3 \div m$	4	$a = \frac{x + ee}{m}$	
$2 \div dee$	5	$a = \frac{b}{dee}$	

4 . 5

$4 \cdot 5$	6	$\frac{x+ee}{m} = \frac{b}{dee}$
$6 \times m$	7	$ee+x = \frac{mb}{dee}$
$7 \times dee$	8	$deeee + xdee = mb$

Here the Equation appears to be of the same Kind with the last, that is quadratic, but not ambiguous. Now by Art. 58.

$9 \div d$	9	$eeee + xee = \frac{mb}{d}$
		And compleating the Square as in the last Question,
10 c □	10	$eeee + xee + \frac{xx}{4} = \frac{mb}{d} + \frac{xx}{4}$
11 ww 2	11	$ee + \frac{x}{2} = \sqrt{\frac{mb}{d} + \frac{xx}{4}}$
$12 - \frac{x}{x}$	12	$ee = \sqrt{\frac{mb}{d} + \frac{xx}{4}} : - \frac{x}{2}$
13 ww 2	13	$e = \sqrt{\sqrt{\frac{mb}{d} + \frac{xx}{4}} : - \frac{x}{2}}$

$$2)\,400000 = mb$$

$$200000 = \frac{mb}{d}$$

$$2500 = \frac{xx}{4}$$

$$202500\ (450 = \sqrt{\frac{mb}{d} + \frac{xx}{4}}$$

$$16 \qquad -\ 50 = -\frac{x}{2}$$

$$85)\,425$$

$$425$$

$$0 \qquad 400\ (20 = \sqrt{\sqrt{\frac{mb}{d} + \frac{xx}{4}} : - \frac{x}{2}} = \text{the}$$

(Number of Miles run by B.

Then by the fourth Step $a = \frac{x+ee}{m} = 100$, the Number of Miles run by A.

This

This Queſtion, as the laſt, may be reſolved in a more ſimple Manner, if we exterminate the unknown Quantity e inſtead of a, thus,

	1	$ma - ee = x$	By the Queſtion, as before
	2	$daee = b$	
$1 + ee$	3	$ma = x + ee$	
$3 - x$	4	$ee = ma - x$	
$2 \div da$	5	$ee = \frac{b}{da}$	
$4 \cdot 5$	6	$ma - x = \frac{b}{da}$	
$6 \times da$	7	$dmaa - dxa = b$	

Dividing by dm the Co-efficient of aa, by Art. 58.

$7 \div dm$	8	$aa - \frac{xa}{m} = \frac{b}{dm}$ for $\frac{dxa}{dm} = \frac{xa}{m}$ the d being rejected as in Diviſion.

Now $\frac{x}{m}$ the Co-efficient of a being divided by 2, is $\frac{x}{2m}$. For making 2 an improper Fraction by the Rule in common Arithmetic, is $\frac{2}{1}$: But by the Rule for Diviſion of Vulgar Fractions in Arithmetic $\frac{2}{1}\Big)\frac{x}{m}\Big(\frac{x}{2m}$. Now ſquaring $\frac{x}{2m}$ is is $\frac{xx}{4mm}$, and adding this to both Sides of the Equation, the *Square is compleated*, by Art. 56.

$8 c \square$	9	$aa - \frac{xa}{m} + \frac{xx}{4mm} = \frac{b}{dm} + \frac{xx}{4mm}$
$9 \text{ ω } 2$	10	$a - \frac{x}{2m} = \sqrt{\frac{b}{dm} + \frac{xx}{4mm}}$
$10 + \frac{x}{2m}$	11	$a = \sqrt{\frac{b}{dm} + \frac{xx}{4mm}} : + \frac{x}{2m} = 100$ (as before.

Then by the fourth or fifth Step we ſhall find $e = 20$, as before.

Queſtion

Queſtion 78. *It is required to find two ſuch Numbers, that the greater being added to the Square of the leſſer, the Sum may be* 19 :

But if the greater is multiplied into the Square of the leſſer, the Product may be 90.

Let a = the greater Number, e = the leſſer Number, $s = 19$, $p = 90$.

	1	$a + ee = s$ } By the Queſtion.
	2	$aee = p$
$1 - ee$	3	$a = s - ee$
$2 \div ee$	4	$a = \frac{p}{ee}$
$3 \cdot 4$	5	$\frac{p}{ee} = s - ee$
$5 \times ee$	6	$p = see - eeee$
		Tranſpoſe $eeee$ to make it affirmative.
$6 + eeee$	7	$eeee + p = see$
$7 - see$	8	$eeee - see + p = 0$
$8 - p$	9	$eeee - see = -p$

Here the Equation not only appears *quadratic*, the Powers of the unknown Quantity e, being the ſame as in the two laſt Queſtions, but it is likewiſe *ambiguous*, for that Side of the Equation which is known is *negative*, *viz.* $-p$.

$9\,c\,\square$	10	$eeee - see + \frac{ss}{4} = \frac{ss}{4} - p$
$10\,\omega\,2$	11	$ee - \frac{s}{2} = \sqrt{\frac{ss}{4} - p}$
$11 + \frac{s}{2}$	12	$ee = \frac{s}{2} \pm \sqrt{\frac{ss}{4} - p}$ the Equation (being *ambiguous*, as above.
$12\,\omega\,2$	13	$e = \sqrt{\frac{s}{2} \pm \sqrt{\frac{ss}{4} - p}}$

That is, by reaſon of the *Ambiguity* of the *Equation*, it may be $e = \sqrt{\frac{s}{2} + \sqrt{\frac{ss}{4} - p}}$, or $e = \sqrt{\frac{s}{2} - \sqrt{\frac{ss}{4} - p}}$

Let us suppose $e = \sqrt{\frac{s}{2} + \sqrt{\frac{ss}{4} - p}}$

$$90.25 = \frac{ss}{4}$$
$$-90. = -p$$
$$\sqrt{.25} = .5 = \sqrt{\frac{ss}{4} - p}$$
$$9.5 = \frac{s}{2}$$
$$10. = \frac{s}{2} + \sqrt{\frac{ss}{4} - p}$$

10) 3.162 nearest $= \sqrt{\frac{s}{2} + \sqrt{\frac{ss}{4} - p}}$ the Value (of e.

```
   9
61) 100
     61
626) 3900
     3756
6322) 14400
      12644
```

Then by the third Step $a = s - ee = 9$, if we take 10 for the Square of e, the square Root of 10 being equal to e.

By trying these Numbers according to the Conditions of the Question, we have

$a + ee = 19$
$aee = 90$ taking 10 for the Square of e, as above.

But because the Value of e is a Fraction which does not terminate, and therefore its exact Value cannot be found, let us try the other Root, *viz.* $e = \sqrt{\frac{s}{2} - \sqrt{\frac{ss}{4} - p}}$

$$90.25 = \frac{ss}{4}$$
$$-90. = -p$$
$$\sqrt{.25} = .5 = \sqrt{\frac{ss}{4} - p}$$

$$\frac{s}{2} = 9.5$$

$$-\sqrt{\frac{ss}{4} - p} = -.5$$

$$\frac{s}{2} - \sqrt{\frac{ss}{4} - p} = 9.$$ extracting the ſquare Root of 9. we have

$$\sqrt{\frac{s}{2} - \sqrt{\frac{ss}{4} - p}} = 3,$$ for the other Value of e exact.

Then by the third Step $a = s - ee = 10.$

And trying theſe two Numbers by the Conditions of the Queſtion, we have

$a + ee = 19$ } As the Queſtion requires, whence the two
$aee \quad = 90$ } Numbers are 10 and 3.

I have been particular in the Arithmetical Work of this Queſtion, that the Learner may ſee the Method of finding both the Values of the unknown Quantity, in any *ambiguous* quadratic Equation, when the unknown Quantity is to the fourth Power.

But in this Queſtion, if we exterminate e inſtead of a, we ſhall have a more ſimple Solution.

	1	$a + ee = s$	By the Queſtion as before.
	2	$aee = p$	
$1 - a$	3	$ee = s - a$	
$2 \div a$	4	$ee = \frac{p}{a}$	
$4 \cdot 3$	5	$\frac{p}{a} = s - a$	
$5 \times a$	6	$p = sa - aa$	
$6 + aa$	7	$aa + p = sa$	
$7 - p$	8	$aa = sa - p$	
$8 - sa$	9	$aa - sa = -p$	

Here the Equation appears *quadratis* and *ambiguous*, as before.

$9 c \square$	10	$aa - sa + \frac{ss}{4} = \frac{ss}{4} - p$

10 ω 2	11	$a - \frac{s}{2} = \sqrt{\frac{ss}{4} - p}$
$11 + \frac{s}{2}$	12	$a = \frac{s}{2} \pm \sqrt{\frac{ss}{4} - p}$

Let us first suppose the Root to be $\frac{s}{2} - \sqrt{\frac{ss}{4} - p}$

$$4)\ 361 = ss$$

$$90.25 = \frac{ss}{4}$$

$$-p = -90.$$

$.25 = \frac{ss}{4} - p$, but the square Root of .25 is .5

whence $.5 = \sqrt{\frac{ss}{4} - p}$

Then $\frac{s}{2} = 9\,5$

$$-\sqrt{\frac{ss}{4} - p} = -0.5$$

$9. = a$, for one of the Roots of the ambiguous Equation, and from this Root, or Value of a, we shall from the third, or fourth Step, find, that e is equal to the square Root of 10, as before; but this being a surd Number, whose Root cannot be exactly extracted, therefore find the other Root, or Value of a, then we have $a = \frac{s}{2} + \sqrt{\frac{ss}{4} - p}$.

$$\frac{s}{2} = 9.5$$

$$+\sqrt{\frac{ss}{4} - p} = .5$$

$10. = a$, the other Root of the ambiguous Equation; then by the third, or fourth Step, we shall find e to be equal to the square Root of 9, which is 3; and these two Numbers 10, and 3, answer the Conditions of the Question.

It may not be improper in this Place to add, that if the Learner meets with any Questions, where the Answers come out

out in Decimal Fractions, he is not from thence to conclude they are not the true Anſwers, as theſe are very frequent and common: But if the Equation is *ambiguous*, it will be proper to find the other Root, which may be free from Fractions; and if this Root anſwers the Conditions of the Queſtion, he has then found the Anſwer compleat: But if the Queſtion will not admit of ſuch an Anſwer, he can then only *approach* to the true Anſwer in continuing his Fractions at Pleaſure; but hitherto I have endeavoured to avoid theſe Circumſtances, as they only fatigue the Learner, and perplex his Mind, inſtead of increaſing his Judgment, or advancing his Knowledge in this Science.

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66. *The Method of reſolving Queſtions, that contain three Equations, and three unknown Quantities.*

FIND the Value of one of the unknown Quantities, in one of the given Equations:

For the ſame unknown Quantity in the other two Equations, write, or put this Value, which exterminates *that unknown Quantity from thoſe two Equations; and reduces the Queſtion to two Equations, and two unknown Quantities, which may be reſolved as the foregoing Queſtions*, by Art. 55. that is,

Find the Value of one of theſe two unknown Quantities, in each of thoſe two Equations, and making theſe two Equations equal to one another, *exterminates* another unknown Quantity, for this laſt Equation will have only one unknown Quantity, which being reduced by the Directions already given, will give the Value of that unknown Quantity in Numbers, from which it will be eaſy to determine the Value of the other two.

To help the Learner in his Choice which to exterminate, if one of the three unknown Quantities is not multiplied, or divided by either of the other two, but theſe are multiplied, or divided by one another, then it will be eaſieſt to find the Value of that unknown Quantity, which is not multiplied, or divided by the others.

Or if one of the unknown Quantities ſhould be to the firſt Power only in all the three given Equations, and the other two are raiſed to ſome higher Power, then it may be eaſieſt to exterminate the unknown Quantity, which is to the firſt Power only.

And if all the three unknown Quantities are only to the firſt Power, and none of them are multiplied or divided by one another, then if one of them has no Co-efficient but *Unity*, and the other two have Co-efficients, it may be eaſieſt to exterminate that unknown Quantity, whoſe Co-efficient is *Unity*.

Theſe Directions may be of Uſe to the Learner, in aſſiſting his Choice which unknown Quantity to exterminate, and a little Care and Attention will help his Judgment in this Part of the Science; I ſhall only juſt mention, that if any particular Difficulties ariſe from the exterminating one unknown Quantity, it may not be improper to make an Eſſay how the Work will proceed, from exterminating ſome other unknown Quantity.

Queſtion 79. *There are three Numbers whoſe Sum is* 18:

The firſt being added to three times the ſecond, from which Sum ſubſtracting twice the third, the Remainder is 9:

But if the firſt is added to four times the third, from which Sum ſubſtracting twice the ſecond, the Remainder is 21. *What are the three Numbers?*

Let a = the firſt Number, e = the ſecond Number, y = the third Number, $b = 18$, $m = 9$, $p = 21$.

$$\begin{array}{r|c|l}
 & 1 & a+e+y=b \\
 & 2 & a+3e-2y=m \\
 & 3 & a+4y-2e=p \\
\hline
1-y & 4 & a+e=b-y \\
4-e & 5 & a=b-y-e
\end{array}\left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\}\text{By the Queſtion.}$$

Having found the Value of a in the firſt Equation, in the room of a in the ſecond and third Equations, put its Value $b-y-e$, thus.

$$\begin{array}{r|c|l}
2 \cdot 5 & 6 & b-y-e+3e-2y=m \\
3 \cdot 5 & 7 & b-y-e+4y-2e=p \\
6 \text{ contracted} & 8 & b-3y+2e=m \\
7 \text{ contracted} & 9 & b+3y-3e=p
\end{array}$$

} Here the Queſtion is reduced to two Equations, and two unknown Quantities, for a is exterminated.

Now

Now find the Value of either y, or e, in each of these two last Equations, and $3\,y$ being in each Equation, find what that is equal to.

$8 + 3y$	10	$b + 2e = m + 3y$
$10 - m$	11	$3y = b + 2e - m$
$9 + 3e$	12	$b + 3y = p + 3e$
$12 - b$	13	$3y = p + 3e - b$
$11 . 13$	14	$p + 3e - b = b + 2e - m$

Here we have an Equation with one unknown Quantity only, which is reduced in the common Manner, thus,

$14 - 2e$	15	$p + e - b = b - m$
$15 + b$	16	$p + e = 2b - m$
$16 - p$	17	$e = 2b - m - p = 6$, then
$13 \div 3$	18	$y = \dfrac{p + 3e - b}{3} = 7$, and
By the fifth Step	19	$a = b - y - e = 5$.

Hence the three Numbers sought are 5. 6. and 7.

PROOF.

$$a + e + y = 18$$
$$a + 3e - 2y = 9$$
$$a + 4y - 2e = 21.$$

Question 80. *Three Men,* A, B, C, *discoursing of their Shillings, found, that if twice* A's *Shillings was added to* B's *Shillings, and from that Sum substracting* C's *Shillings, there remains* 15:

And if B's *Shillings was added to three times* C's *Shillings, and from that Sum substracting* A's *Shillings, there remains* 31:

But if six times A's *Shillings was added to four times* C's *Shillings, and this Sum added to* B's *Shillings, the Sum was* 97. *How many Shillings had each Person?*

Let $a =$ the Number of Shillings of A, $e =$ those of B, and $y =$ those of C, $b = 15$, $d = 31$, $m = 97$.

1	$2a + e - y = b$	By the Question.
2	$e + 3y - a = d$	
3	$6a + 4y + e = m$	

Because

Becauſe e has no Co-efficient but *Unity*, begin with finding the Value of e, as being the moſt ſimple.

$1 + y$	4	$2a + e = b + y$
$4 - 2a$	5	$e = b + y - 2a$

Now in the ſecond and third Equations, in the room of e, put its Value, or $b + y - 2a$, as in the laſt Queſtion.

$2 . 5$	6	$b + y - 2a + 3y - a = d$
$3 . 5$	7	$b + y - 2a + 6a + 4y = m$

Here the Queſtion is reduced to two Equations, and two unknown Quantities, e being exterminated, and therefore proceeding, as in the laſt Queſtion,

6 contracted	8	$b + 4y - 3a = d$
7 contracted	9	$4a + 5y + b = m$

Now find the Value of either of the unknown Quantities in both theſe Equations: To find the Value of y,

$8 + 3a$	10	$b + 4y = d + 3a$
$10 - b$	11	$4y = d + 3a - b$
$11 \div 4$	12	$y = \frac{d + 3a - b}{4}$
$9 - b$	13	$4a + 5y = m - b$
$13 - 4a$	14	$5y = m - b - 4a$
$14 \div 5$	15	$y = \frac{m - b - 4a}{5}$
$12 . 15$	16	$\frac{d + 3a - b}{4} = \frac{m - b - 4a}{5}$

Here we have an Equation with only the unknown Quantity a.

16×4	17	$d + 3a - b = \frac{4m - 4b - 16a}{5}$
17×5	18	$5d + 15a - 5b = 4m - 4b - 16a$
$18 + 16a$	19	$5d + 31a - 5b = 4m - 4b$
$19 + 5b$	20	$5d + 31a = 4m + b$
$20 - 5d$	21	$31a = 4m + b - 5d$
$21 \div 31$	22	$a = \frac{4m + b - 5d}{31} = 8$, then

By

By the 12th Step	23	$y = \frac{d + 3a - b}{4} = 10$, and
By the fifth Step	24	$e = b + y - 2a = 9$

PROOF.

$$2a + e - y = 15$$
$$e + 3y - a = 31$$
$$6a + 4y + e = 97$$

I have done these two Questions without putting Letters for the given Numbers, it being more easy and familiar; but now to do the last universally, let us put Letters for the Numbers 2 . 3 . 6 and 4 which are given in the Question, and comparing the former Operation with the following, may render it more easy; but if the Learner finds this too perplexing, he may neglect it, and proceed to the next.

Let a . e and y be the three unknown Numbers as before, and $x = 2$, $z = 3$, $s = 6$, $p = 4$, then,

1	$xa + e - y = b$	By the Question.
2	$e + zy - a = d$	
3	$sa + py + e = m$	

Because e has no *specious* Co-efficient in either of the given Equations, find the Value of e.

$1 + y$	4	$xa + e = b + y$
$4 - xa$	5	$e = b + y - xa$

Now in the second and third Equations, in the room of e put its Value, or $b + y - xa$,

2 . 5	6	$b + y - xa + zy - a = d$
3 . 5	7	$sa + py + b + y - xa = m$

Here the Question is reduced to two Equations, and two unknown Quantities, e being exterminated; but because of the specious Co-efficients we cannot contract them as before: Now find the Value of y, in both these Equations.

$6 + a$	8	$b + y - xa + zy = d + a$
$8 + xa$	9	$b + y + zy = d + a + xa$
$9 - b$	10	$zy + y = d + a + xa - b$
$10 \div z + 1$	11	$y = \frac{d + a + xa - b}{z + 1}$ the Co-efficients of y being ($z + 1$
$7 + xa$	12	$sa + py + b + y = m + xa$
$12 - b$	13	$sa + py + y = m + xa - b$
$13 - sa$	14	$py + y = m + xa - b - sa$
$14 \div p + 1$	15	$y = \frac{m + xa - b - sa}{p + 1}$ the Co-efficients of y being ($p + 1$
$11 . 15$	16	$\frac{d + a + xa - b}{z + 1} = \frac{m + xa - b - sa}{p + 1}$ an Equation (with only the unknown Quantiy a.
$16 \times z + 1$	17	$d + a + xa - b = \frac{zm + zxa - zb - zsa + m + xa - b - sa}{p + 1}$
$17 \times p + 1$	18	$pd + pa + pxa - pb + d + a + xa - b = zm + zxa - zb - zsa + m + xa - b - sa$

The Learner may think these Multiplications discouraging, though perhaps they are not so perplexing as he may imagine, for at the seventeenth Step where $m + xa - b - sa$ is $\times z + 1$, put down the Product of it by z first, which is $zm + zxa - zb - zsa$, after which he need only write $m + xa - b - sa$, the next Part of the Multiplier being *Unity*; or, if it had been another Letter, it had been no more than repeating the *Multiplicand*, with the *multiplying Letter* joined to each of its Quantities, placing them one after another, taking due Care of the Signs by the Rules for Multiplication.

In the same Manner he will find the eighteenth Step multiplied, and a little Attention will familiarize the Operation; but if there is any Difficulty in multiplying these compound Quantities, the Learner may set them down one under the other, and multiply them in the usual Manner.

$18 - xa$	19	$pd + pa + pxa - pb + d + a - b = zm + zxa - zb - zsa + m - b - sa$
$19 + b$	20	$pd + pa + pxa - pb + d + a = zm + zxa - zb - zsa + m - sa$

Now transpose all the unknown Quantities to one Side of the Equation, and all the known ones to the other Side of the Equation.

$20 - zxa$	21	$pd + pa + pxa - pb + d + a - zxa = zm - zb - zsa + m - sa$
$21 + zsa$	22	$pd + pa + pxa - pb + d + a + zsa - zxa = zm - zb + m - sa$
$22 + sa$	23	$pd + pa + pxa - pb + d + a + zsa - zxa + sa = zm - zb + m$
$23 - pd$	24	$pa + pxa - pb + d + a + zsa - zxa + sa = zm - zb + m - pd$
$24 + pb$	25	$pa + pxa + d + a + zsa - zxa + sa = zm - zb + m - pd + pb$
$25 - d$	26	$pa + pxa + a + zsa - zxa + sa = zm - zb + m - pd + pb - d$
$26 \div$	27	$a = \frac{zm - zb + m - pd + pb - d}{p + px + 1 + zs - zx + s} = 8$, the Divisor is the *Co-efficients* of a, connected by their Signs.
Then by 11th Step	28	$y = \frac{d + a + xa - b}{z + 1} = 10$
And by 5th Step	29	$e = b + y - xa = 9$

Question 81. *There are three Travellers,* A, B, C, *who have travelled in all* 62 *Miles:*

But if the Miles A *travelled is multiplied by* 2, *and added to the Miles* B *travelled multiplied by* 3, *this Sum is equal to the Miles* C *travelled multiplied by* 17:

And if 4 *times the Miles* C *travelled, is added to the Miles* B *travelled multiplied by* 2, *this Sum is equal to the Miles travelled by* A. *To find the Miles each travelled?*

Let $a =$ the Miles travelled by A, $e =$ the Miles travelled by B, $y =$ the Miles travelled by C; $p = 62$, $b = 2$, $d = 3$, $m = 17$, $x = 4$, and 2 being in the Question before, put no new Letter for it.

1	$a + e + y = p$	By the Question.
2	$ba + de = my$	
3	$xy + be = a$	

Because a seems to be in as simple Terms as any in the three given Equations, and having its Value already by the third Equation, therefore for a in the first and second Equation write its Value $xy + be$ at the third Equation, which exterminates a.

1 . 3	4	$xy+be+e+y=p$
2 . 3	5	$bxy+bbe+de=my$

Here the Queſtion is reduced to two Equations, and two unknown Quantities, then proceed, as before, to find the Value of either y or e, in each of theſe Equations, as ſuppoſe y.

$4-be$	6	$xy+e+y=p-be$
$6-e$	7	$xy+y=p-be-e$
$7\div x+1$	8	$y=\frac{p-be-e}{x+1}$ the Co-efficients of y being $x+1$.
$5-bxy$	9	$my-bxy=bbe+de$
$9\div m-bx$	10	$y=\frac{bbe+de}{m-bx}$ the Co-efficients of y being $m-bx$.
8 . 10	11	$\frac{bbe+de}{m-bx}=\frac{p-be-e}{x+1}$ an Equation with only (one unknown Quantity.
$11\times m-xb$	12	$bbe+de=\frac{mp-mbe-me-pbx+bbxe+bxe}{x+1}$
$12\times x+1$	13	$xbbe+xde+bbe+de=mp-mbe-me-pbx+bbxe+bxe$
$13-xbbe$	14	$xde+bbe+de=mp-mbe-me-pbx+bxe$
		Now bring the Terms that have the unknown Quantity, to one Side of the Equation.
$14+mbe$	15	$xde+bbe+de+mbe=mp-me-pbx+bxe$
$15+me$	16	$xde+bbe+de+mbe+me=mp-pbx+bxe$
$16-bxe$	17	$xde+bbe+de+mbe+me-bxe=mp-pbx$
$17\div$	18	$e=\frac{mp-pbx}{xd+bb+d+mb+m-bx}=9$, the Diviſor is the Co-efficients of e, connected by their Signs.
By Step 8, or 10.	19	$y=7$
By Step 3.	20	$a=46$

Queſtion 82. *Three Men,* A, B, C, *diſcourſing of their Shillings, found, that* A's *Shillings added to* C's *Shillings, the Sum was double* B's *Shillings:*

And A's *Shillings added to three times* B's *Shillings, from which Sum ſubſtracting* C's *Shillings, there remained* 13 *Shillings:*

But if A's *Shillings was added to the Product of* B's *and* C's *Shillings, the Sum was* 34. *How many Shillings had each Perſon?*

Let

Let a = A's Shillings, e = B's Shillings, y = C's Shillings, $b = 13$, $d = 34$.

	1	$a+y=2e$	By the Question.
	2	$a+3e-y=b$	
	3	$a+ey=d$	
$1-y$	4	$a=2e-y$	
$2 \cdot 4$	5	$2e-y+3e-y=b$	Here the Question is reduced to two Equations and two unknown Quantities, a being exterminated.
$3 \cdot 4$	6	$2e-y+ey=d$	
5 contracted	7	$5e-2y=b$	

Now find the Value of e, or y, in the sixth and seventh Equations, suppose e.

$6+y$	8	$2e+ey=d+y$	
$8 \div \overline{2+y}$	9	$e=\dfrac{d+y}{2+y}$	
$7+2y$	10	$5e=b+2y$	
$10 \div 5$	11	$e=\dfrac{b+2y}{5}$	
$9 \cdot 11$	12	$\dfrac{b+2y}{5}=\dfrac{d+y}{2+y}$,	an Equation with only one unknown Quantity.
$12 \times \overline{2+y}$	13	$\dfrac{2b+4y+by+2yy}{5}=d+y$	
13×5	14	$2b+4y+by+2yy=5d+5y$	

Now bring all the Quantities that have y, to one Side of the Equation.

$14-5y$	15	$2b-y+by+2yy=5d$
$15-2b$	16	$2yy+by-y=5d-2b$

Here the Equation appears *quadratic*, the unknown Quantity being to the second and first Power only, but is not ambiguous, $5d$ being greater than $2b$; then by Art. 58, divide by the Co-efficient of yy.

$16 \div \overline{2}$	17	$yy+\dfrac{by-y}{2}=\dfrac{5d-2b}{2}$

The Work being now prepared for *compleating the Square*, becauſe the Co-efficient of y is $\frac{b-1}{2}$, to avoid the Trouble of dividing this Fraction by 2, and ſquaring the Quotient, ſubſtitute by Art. 57. $x=\frac{b-1}{2}=6$.

Then	18	$yy+xy=\frac{5d-2b}{2}$
$18\,c\,\square$	19	$yy+xy+\frac{xx}{4}=\frac{5d-2b}{2}+\frac{xx}{4}$
19 ɯ 2	20	$y+\frac{x}{2}=\sqrt{\frac{5d-2b}{2}+\frac{xx}{4}}$
$20-\frac{x}{2}$	21	$y=\sqrt{\frac{5d-2b}{2}+\frac{xx}{4}}:-\frac{x}{2}=6$, (C's Shillings.
By the 9th, or 11th Steps	22	$e=5$, B's Shillings.
By the 4th Step	23	$a=4$, A's Shillings.

Queſtion 83. *Three young Gentlemen,* A, B, C, *having been at the Gaming-Tables, from comparing their Loſſes, found, that if from twice the Pounds* A *loſt, was ſubſtracted the Pounds* B *loſt, there remained the Pounds* C *loſt:*

And that the Pounds A *loſt, added to the Pounds* B *loſt, and this Sum added to twice the Pounds* C *loſt, the Sum was* 19 *Pounds:*

But if to the Product of A's *and* C's *Loſs, there is added* B's *Loſs, the Sum is* 26 *Pounds. How much did each Perſon loſe?*

Let $a=$ A's Loſs, $e=$ B's Loſs, $y=$ C's Loſs, $d=19$, $b=26$.

1	$2a-e=y$	By the Queſtion.
2	$a+e+2y=d$	
3	$ay+e=b$	

Becauſe e ſeems to be in the moſt ſimple Terms, therefore find its Value.

$1+e$	4	$2a=y+e$
$4-y$	5	$e=2a-y$

2 . 5	6	$a+2a-y+2y=d$
3 . 5	7	$ay+2a-y=b$
6 contracted	8	$3a+y=d$

The Question is here reduced to two Equations and two unknown Quantities, for e is exterminated.

Find the Value of a, or y, in the seventh and eighth Equations.

$7-2a$	9	$ay-y=b-2a$
$9 \div a-1$	10	$y=\dfrac{b-2a}{a-1}$
$8-3a$	11	$y=d-3a$
10 . 11	12	$\dfrac{b-2a}{a-1}=d-3a$
$12 \times a-1$	13	$b-2a=da-3aa-d+3a$

Now bring all the Quantities that have a on one Side of the Equation, observing to have the highest Power of a *affirmative.*

$13+3aa$	14	$3aa+b-2a=da-d+3a$
$14-3a$	15	$3aa+b-5a=da-d$
$15-da$	16	$3aa+b-5a-da=-d$
$16-b$	17	$3aa-5a-da=-d-b$

Here the Equation appears both *quadratic* and *ambiguous*, for the unknown Quantity is to the second and first Power only, and it is *ambiguous*, because $-d-b$ the Side of the Equation which is known, is *negative*; dividing by the Co-efficient of aa, as in the last Question,

$17 \div \overline{3}$	18	$aa-\dfrac{5a-da}{3}=\dfrac{-d-b}{3}$

The Work being now prepared for *compleating the Square*, substitute $-x=\dfrac{-5-d}{3}=-8$ the Co-efficients of a, as in the last Example.

Then	19	$aa-xa=\dfrac{-d-b}{3}$
$18\ c\ \square$	20	$aa-xa+\dfrac{xx}{4}=\dfrac{-d-b}{3}+\dfrac{xx}{4}$

19 w

19 ѡ 2	21	$a - \frac{x}{2} = \sqrt{\frac{-d-b}{3} + \frac{xx}{4}}$
$20 + \frac{x}{2}$	22	$a = \frac{x}{2} \pm \sqrt{\frac{-d-b}{3} + \frac{xx}{4}} = 4,$ $(\pm 1 = 3,$ or $5.$

For the Practice of the Learner, let us ſuppoſe $a = 3$
Then by the tenth, or eleventh Steps - $y = 10$
And by the fifth Step - - $e = 6 - 10$
$= -4$, which is an Impoſſibility, that e an *affirmative* Quantity, can be equal to a *negative* 4.
Now let us ſuppoſe - - $a = 5$
Then by the tenth, or eleventh Steps - $y = 4$
And by the fifth Step - - - - - $e = 6$

$$2a - e = y$$
$$a + e + 2y = 19$$
$$ay + e = 26$$

Then $2a - e = y$, $a + e + 2y = 19$, $ay + e = 26$

And theſe three Numbers anſwering the Conditions of the Queſtion, are the true Numbers ſought; from hence the young *Analyſt* may obſerve, that in *quadratic ambiguous* Equations, if one of the Roots of the unknown Quantity does not anſwer the Conditions of the Queſtion, he ſhould find the other Root, and try that, before he concludes his Work erroneous.

I ſhall now ſhow the Learner the excellent Method of reſolving all Equations, be their Powers never ſo high, by the univerſal Method of *Converging Series*.

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67. *The Reſolution of Adfected Equations, by the univerſal Method of Converging Series.*

CASE I.

Ex. 1. SUPPOSE there was given $aaa + a = 9282$, to find a

Then ſuppoſe, or imagine a to be - - 20
Conſequently the Cube of a, or aaa, is - 8000

These being added together, becauſe it is $aaa + a$ in the given Equation, the Sum is - - } 8020

Hence a muſt be more than 20, for if that had been the true Root, the Cube of 20 added to its firſt Power, or 20, muſt have been equal to 9282 the given Number, for theſe are the ſame Powers of a as in the given Equation; but that Sum being only 8020, which being leſs than 9282, the Value of a muſt be more than 20. To find how much that is?

Let $r = 20$, and for what 20 wants of the true Number or Root, put e:

Then will $r + e = a$, or the true Root of the Equation, hence by determining what e is, we find the Number that is to be added to r or 20, which Sum will be the Root of the given adfected Equation, to do which, put down,

	1	$r + e = a$

Now raiſe this Equation to the third Power, becauſe we have aaa in the given Equation.

1 ⓖ 3	2	$rrr + 3rre + 3ree + eee = aaa$

Add the firſt and ſecond Equations together, becauſe in the given Equation it is $aaa + a$.

$1 + 2$	3	$rrr + 3rre + 3ree + eee + r + e = aaa + a$
		But from the given Equation
it is	4	$aaa + a = 9282$
$3 \cdot 4$	5	$rrr + 3rre + 3ree + eee + r + e = 9282$ each Equation being equal to $aaa + a$.
		Putting this Equation into Numbers, and rejecting the Powers of e above ee.
5 in Numbers	6	$8000 + 1200e + 60ee + 20 + e = 9282$
That is	7	$8020 + 1201e + 60ee = 9282$
$7 - 8020$	8	$1201e + 60ee = 1262$
		Divide by the Co-efficient of ee, and we have,
$7 \div 60$	9	$20.016e + ee = 21.0333$ *ad infinitum.*
		Dividing by $20.016 + e$, that is, by the Co-efficient of e *plus* e, and we have,
$9 \div 20.016 + e$	10	$e = \dfrac{21.03333}{20.016 + e}$

In Numbers thus:

$$
\begin{array}{l}
20.016)\ 21.0333\ (1 = e \\
\underline{\quad 1 \quad} \qquad 21016 \\
21.016 \quad \overline{\qquad\qquad} \\
\qquad\qquad 17 \text{ the Remainder}
\end{array}
$$

(being very ſmall reject it.

For the Reaſon of adding the Quotient Figure, or 1, to the Diviſor, ſee Article 68, in the next Page.

By

By this it appears that $e = 1$, that is, 1 is to be added to the first supposed Number 20, which Sum is to be the Value of a, or the Root of the given adfected Equation.

$$\begin{array}{rl} \text{We assumed } r = & 20 \\ \text{And found } e = & 1 \\ \hline r + e = & 21 = a \end{array}$$

To try whether 21 is the true Root, raise it to the several Powers of a in the given Equation.

$$\begin{array}{rr} a = & 21 \\ a = & 21 \\ \hline & 21 \\ & 42 \\ \hline aa = & 441 \\ a = & 21 \\ \hline & 441 \\ & 882 \\ \hline aaa = & 9261 \\ a = & 21 \\ \hline \end{array}$$

Then $aaa + a = 9282$, which being the same with the Number in the given Equation, it appears that $a = 21$.

68. By reviewing the Operation, the Learner may observe,

First, That we supposed a Number for the true Root, which upon Trial was found less than the true Root.

Secondly, For that Deficiency or Want, we put e, or any other Letter.

Thirdly, By connecting $r =$ the Number first supposed to be the Root, with e by the Sign $+$, we have $r + e$ for the true Root, r being a known Quantity, and e the unknown Quantity.

Fourthly, We raise $r + e$ to the several Powers of the unknown Quantity, that are in the given adfected Equation.

Fifthly, Then we *add* these several Equations together, rejecting all the Powers of e, or of the unknown Quantity above the *Square*, for in the given Equation all the Powers of the unknown Quantity have the Sign $+$, but when any of these have the Sign $-$, then their respective Equations must be *substracted*, as at Step 5, Example 3, Page 238.

Sixthly, By these Means we have an Equation in the Terms of r and e, equal to the given Equation.

Seventhly,

Seventhly, This Equation is put into Numbers, r being a known Quantity, and the less absolute Number is transposed to the Side of the Equation of the greater absolute Number, and substracted from it.

Eighthly, After this, the Equation is divided by the Co-efficient of the Square of e, or the unknown Quantity.

Ninthly, This last Equation is divided by the Co-efficient of e *plus* e, which leaves e on one Side of the Equation by itself.

Tenthly, In the Arithmetical Work, because the last Divisor consists of a Number *plus* e, therefore, as the Quotient Figure is found, it is added to the Divisor to make it compleat; and if the numerical Operation had been continued to more Places of Figures in the Quotient, then the Quotient Figure must be twice added, once when it is found, and once at the next Step in the Division, as in the next Page.

Eleventhly, The Quotient thus found being the Value of e, or the unknown Quantity, it is added to the Number first supposed to be the Root of the Equation, which is represented by r, and this Sum is supposed to be the Root required.

This Operation to find e is the same as the common Method of finding the unknown Quantity, till we come to the tenth Step, where the unknown Quantity making Part of the Divisor, it is carried to the other Side of the Equation, and the Divisor being a known Number $+ e$, the Quotient as it is found is added to the Divisor, to make it compleat, as before-mentioned.

But if this Number should not be the true Root, the Operation must be repeated, making the Number thus found $= r$, and at the second Operation, the Work in any common Case will be sufficiently exact: And from the Repetition of the Operation, whereby we approach nearer and nearer to the true Root, this Method is called the Method of *Converging Series*, or of *Approximation*.

Example 2. Suppose $aaa + aa + a = 42997850$, to find a.

Suppose a to be - - -	300
Then the Cube of a, or aaa is - -	27000000
And the Square of a, or aa is - -	90000
These being added, because it is $aaa + aa + a$ in the given Equation, the Sum is - -	27090300

But 42997850, the given Number, is greater than 27090300, therefore the Root must be more than 300.

Now let $r = 300$, and $e =$ what 300 wants of the true Root.

Then	1	$r+e=a$
$1 ⓪ 3$	2	$rrr+3rre+3ree+eee=aaa$ } According to Particular 4, Art. 68.
$1 ⓪ 2$	3	$rr+2re+ee=aa$
$1+2+3$	4	$r+e+rrr+3rre+3ree+rr+2re+ee=aaa+aa+a$, by Particulars 5 and 6, Art. 68.
But	5	$aaa+aa+a=42997850$, from the given Equation.
$4 \cdot 5$	6	$r+e+rrr+3rre+3ree+rr+2re+ee=42997850$
6 in Numbers	7	$300+e+27000000+270000e+900ee+90000+600e+ee=42997850$
That is	8	$27090300+270601e+901ee=42997850$
$8-27090300$	9	$270601e+901ee=15907550$
$9 \div 901$	10	$300.334e+ee=17655.43$ from Particular 8, Art. 68.
$13 \div 300.334+e$	11	$e=\dfrac{17655.43}{300.334+e}$, from Particular 9, (Art. 68.

```
         300.334) 17655.43 (50.34 = e,
           5
        -------
Divisor 350.334   1751670
          50.3    -------
        -------    1387300
Divisor 400.634    1201902
           .34    -------
        -------     1853980
Divisor 400.974     1603896
                    -------
                     250084
```

the first Figure being in the Place of *Tens*, place the 5 under the Place of Tens in the Divisor; and this the Reader is to observe, to place the Quotient Figures he adds to the Divisor, under those of the same Denomination.

$r = 300$
$e = 50.34$

$r+e = 350.34 = a$, hence I suppose the Root of the given Equation is 350.34 but to try it, raise 350.34 to the several Powers of a in the given Eqnation.

$a =$

$$\begin{array}{r}
a = 350.34 \\
a = 350.34 \\
\hline
140136 \\
105102 \\
1751700 \\
105102 \\
\hline
aa = 122738.1156 \\
a = 350.34 \\
\hline
4909524624 \\
3682143468 \\
61369057800 \\
3682143468 \\
\hline
aaa = 43000071.419304 \\
aa = 122738.1156 \\
a = 350.34 \\
\hline
\end{array}$$

Then $aaa + aa + a = 43123159.874904$ which being greater than 42997850, the Root cannot be ſo much as 350.34 and this leads us to explain the Method of finding the true Root, when the Number aſſumed is greater than the Root required, or,

CASE 2.

Let us take the laſt *Example*, *viz.* $aaa + aa + a = 42997850$.

And ſuppoſe the Root to be 350.34 which we know is too much by the laſt Operation:

Now put $e =$ the Number to be ſubſtracted from 350.34 ſuppoſing $r = 350.34$ and to r connecting e by the Sign —, we have,

	1	$r - e = a$, or the true Root	By Particular 4. Art. 68.
1 ☉ 3	2	$rrr - 3rre + 3ree - eee = aaa$	
1 ☉ 2	3	$rr - 2re + ee = aa$	

Now collect theſe three Equations by Art. 68, Particulars 5 and 6, and rejecting eee, we have,

1 + 2 + 3	4	$r - e + rrr - 3rre + 3ree + rr - 2re + ee = aaa + aa + a$
But	5	$aaa + aa + a = 42997850$, from the given Equation.
4 . 5	6	$r - e + rrr - 3rre + 3ree + rr - 2re + ee = 42997850$

6 in Num.	7	$350.34 - e + 43000071.42 - 368214.3468e + 1051.02ee + 122738.1156 - 700.68e + ee = 42997850$
That is	8	$43123159.8756 - 368916.0268e + 1052.02ee = 42997850$ Now tranſpoſe 42997850, it being leſs than 43123159.8765
8 −	9	$125309.8756 - 368916\,0268e + 1052.02ee = 0$, for one Side of the Equation being ſubſtracted from the other Side, the Remainder muſt be nothing, as both Sides of the Equation are equal. Now tranſpoſe the ſeveral Quantities which contain e to the other Side of the Equation.
9 +	10	$125309.8756 + 1052.02ee = 368916.0268e$
10 −	11	$368916.0268e - 1052.02ee = 125309.8756$ Here dividing by the Co-efficient of ee as before,
11 ÷	12	$350.673e - ee = 119.1135$ But now divide by the Co-efficient of e *minus e.*
12 ÷	13	$e = \dfrac{119.1135}{350.673 - e}$

In Numbers:

	350.673)	119.1135 (.34
	− .3	
Diviſor	350.373	1051119
	− .34	
		1400160
Diviſor	350.033	1400132
		28

Having thus determined e to be .34 it muſt now be ſubſtracted from r, becauſe it was aſſumed $r - e =$ the true Root.

But r was ſuppoſed $= 350.34$

$-e$, which we have found $= .34$

Hence $r - e = 350. = a$, the Root of the given adfected

adfected Equation, which is proved by raising 350. to the several Powers of a in the given Equation,

$$\begin{aligned} \text{Thus, } aaa &= 42875000 \\ aa &= 122500 \\ a &= 350 \end{aligned}$$

Consequently $aaa + aa + a = 42997850$ which being the same Number as in the given Equation, it shows that a is exactly equal to 350.

In the above Operation, at the thirteenth Step, the Learner may observe, that the Divisor is $350.673 - e$, therefore here, as the Quotient Figure is found, we *substract* it from the Part of the Divisor 350.673 to have the Divisor compleat, which is likewise done twice, once before the Division is made at that Figure, and once afterwards: But in the first *Case*, when r is assumed too little, then the Quotient Figure is added, as at Particular 10, Art. 68. the Sign then being contrary to what it is now.

The Learner may further observe, that by this second Operation, we have found the *true Root*, whereas by the first Operation it was .34 too much, and therefore if the true Root does not come out at the first Operation, make a second Operation, supposing the Number found at the first Operation to be r, and call it $r + e$, or $r - e$, for the true Root as the Occasion requires, that is, as the Number at the first Operation is either greater or lesser than the true Root; which second Operation will give the true Root very near, and near enough for any common Case, though if the Arithmetical Divisions were continued, as they will not terminate, do not give the true Root exactly, as in the Division of those Decimal Fractions which never terminate; in such Divisions we leave off when the Quotient is to a sufficient Degree of Exactness, so the same is done here when we are near enough the Truth; and in common Cases, two or three Places of Decimal Fractions are sufficient, and according as they happen the true Root is sometimes found; and in general, continue the Division, at the *second Operation*, to as many Places of Decimal Fractions as are in the Number found in the *first Operation:* And after the Number found at the second Operation is added to, or substracted from the Number found at the first Operation, if there is a very small Fraction you may reject it; but if the Fraction should be very near an *Unit*, then take 1 for it, which add to the Integers, and try whether the whole Number thus found is not the true Root. In Arithmetical Questions, whose Answers are often

often in whole Numbers, this Caution may help the Learner to chuse the true Root exactly.

The Reason why this Method does not *absolutely* give the true Root is the *arbitrary* rejecting all the Powers of e above ee.

Example 3. Admit $aaa - aa + a = 46526760$, to find a.

Now suppose $a = 400$.

Then aaa is - - -	64000000
And aa is 160000, which must be *substracted* because it is $-aa$ in the given Equation -	160000
	63840000
To which adding a, or 400, it being $+a$ in the given Equation - - -	400
Hence $aaa - aa + a$ is . -	63840400

Which exceeding 46526760 the Number in the given Equation, a must be less than 400.

Then let $r = 400$, $e =$ the Number that 400 is too much, which being the *second Case*, Page 235.

Hence	1	$r - e = a$
1 Ⓒ 3	2	$rrr - 3rre + 3ree - eee = aaa$
1 Ⓒ 2	3	$rr - 2re + ee = aa$

Because in the given Equation the Quantities aaa and a are *affirmative*, therefore add the first and second Equations together.

1 + 2	4	$r - e + rrr - 3rre + 3ree - eee = aaa + a$

Because in the given Equation it is $-aa$, therefore *substract* the third Equation from the fourth, or Sum of the first and second Equations. And here the Reader is to observe, that if in the given adfected Equation, any Powers of the unknown Quantity have the Sign $-$, the Equation which arises from involving $r - e$ to such Powers, is to be *substracted* instead of being added.

4 − 3	5	$r - e + rrr - 3rre + 3ree - eee - rr + 2re - ee = aaa - aa + a$
But	6	$aaa - aa + a = 46526760$, by the given Equation.
5 . 6	7	$r - e + rrr - 3rre + 3ree - eee - rr + 2re - ee = 46526760$

 Putting

Putting this Equation in Numbers, and rejecting all the Powers of e above ee.

7 in Numbers	8	$400 - e + 64000000 - 480000e + 1200ee - 160000 + 800e - ee = 46526760$
8 contracted	9	$63840400 + 1199ee - 479201e = 46526760$
		Transpose 46526760 it being less than 63840400
9 −	10	$17313640 + 1199ee - 479201e = 0$
		Now transpose the Quantities that have e, to the other Side of the Equation.
10 +	11	$479201e = 17313640 + 1199ee$
11 −	12	$479201e - 1199ee = 17313640$
		Dividing by the Co-efficient of ee,
12 ÷	13	$399.66e - ee = 14440.06$
		Now dividing by the Co-efficient of e *minus* e.
13 ÷	14	$e = \dfrac{14440.06}{399.66 - e}$

In Numbers thus:

```
          399.66)  14440.06  (40.16 = e.
         — 40.
         --------
Divisor  359.66   143864
         — 40.1   ------
         -------       53660
Divisor  319.56        31956
         —   .16       ------
         -------       217040
Divisor  319.40        191640
                       ------
                        25400
```

Now $r = 400$
$-e = 40.16$

$r - e = 359.84 = a$, and to try if this is the true Root, raise it to the several Powers of a, in the given Equation.

$a =$

$$
\begin{array}{r}
a = 359.84 \\
a = 359.84 \\
\hline
143936 \\
287872 \\
323856 \\
179920 \\
107952 \\
\hline
aa = 129484.8256 \\
a = 359.84 \\
\hline
5179393024 \\
10358786048 \\
11653634304 \\
6474241280 \\
3884544768 \\
\hline
aaa = 46593819.643904 \\
-aa = 129484.8256 \\
\hline
\text{Remains } 46464334.818304 \\
+a = 359.84 \\
\hline
\end{array}
$$

Sum, or $aaa - aa + a = 46464694.658304$ which being lefs than 46526760 the Number in the given Equation, the Root or a muft be more than 359.84.

Therefore, for a fecond Operation, fuppofe $r = 359.84$ and $e =$ what it wants of the true Root, then it being $r + e = a$, it is now the *firft Cafe*, Page 230.

Therefore	1	$r + e = a$
1 ⓖ 3	2	$rrr + 3rre + 3ree + eee = aaa$
1 ⓖ 2	3	$rr + 2re + ee = aa$
		Add the firft and fecond Equation together, becaufe in the given Equation it is $a + aaa$.
1 + 2	4	$rrr + 3rre + 3ree + eee + r + e = aaa + a$
		From this Equation fubftract the third Equation, becaufe it is $-aa$ in the given Equation.
4 − 3	5	$rrr + 3rre + 3ree + eee + r + e - rr - 2re - ee = aaa - aa + a$
But	6	$aaa - aa + a = 46526760$ by the given Equation.
5 . 6	7	$rrr + 3rre + 3ree + eee + r + e - rr - 2re - ee = 46526760$

Put

		Put this Equation in Numbers, and reject the Powers of e, above ee.
7 in Numbers	8	$46593819.644 + 388454.4768\,e + 1079.52ee + 359.84 + e - 129484.8256 - 719.68e - ee = 46526760$
8 contracted	9	$46464694.6584 + 387735.7968\,e + 1078.52\,ee = 46526760$
9 —	10	$387735.7968\,e + 1078.52\,ee = 62065.3416$
		Dividing by the Co-efficient of ee.
10 ÷	11	$359.5\,e + ee = 57.547$
		Now dividing by the Co-efficient of e *plus e.*
11 ÷	12	$e = \dfrac{57.547}{359.5 + e}$

In Numbers thus:

$$\begin{array}{lrl}
 & 359.5) & 57.547\ (.16 = e. \\
 & + \quad .1 & \\
\hline
\text{Divisor} & 359.6 & 3596 \\
 & .16 & \overline{215870} \\
\hline
\text{Divisor} & 359.76 & 215856 \\
 & & \overline{\quad 14}
\end{array}$$

The Reader will observe that in this Division, I have taken at once two Figures from the Dividend, *viz.* 70, because in adding the .16 to the Divisor, the Number of Places there is increased by one, therefore I take one Figure more from the Dividend than is usual; which is recommended to the Reader's Attention, as he may again meet with the same Case.

$$\begin{array}{rl}
\text{Now } r = & 359.84 \\
+\, e = & .16 \\
\hline
r + e = & 360.00 = a,
\end{array}$$

which will be found to be the true Root, by involving it to the several Powers of a in the given Equation, and adding or substracting them according as those Powers of a are there connected by the Signs + or —.

It may be proper to inform the Learner, that the nearer the Number is taken to the true Root, the nearer the Operation will come to the Truth, and therefore after he has tried the *first Supposition*, if he thinks he can make a second Supposition nearer the Truth, it will be right to do it, which perhaps

may bring out the Root so near at the first Operation, that it may save him the Trouble of making a second Operation. Thus,

If $aaa + aa + a = 4942070$, to find a.

Suppose a to be 100.

Then the Cube of a, or aaa is - -	1000000
And the Square of a, or aa is - - -	10000
And a is - - - -	100
	1010100
The Number in the given Equation is -	4942070
If we suppose $a = 100$, then the Sum of its several Powers are - - -	1010100
Difference *wanting* - - -	3931970

Now let us make a second Supposition thus,

If $a = 200$,

Then the Cube of a, or aaa is - -	8000000
And the Square of a, or aa is - -	40000
And a is - - - - -	200
Sum of the several Powers of a, if a is 200 -	8040200
The Number in the given Equation - -	4942070
Difference *over*	3098130

For a *third* Supposition, suppose it 160 and try with that, and if it be *less than just*, it must be $r = 160$ and $r + e = a$; if 160 be too much or *more than just*, then it must be $r - e = a$.

When there are two Suppositions made, one being *more than just*, and the other *less than just*, it may be convenient to make a third Supposition between the two, and proceed by *Case* 1 or 2, according as the supposed Number is *more* or *less than just*.

Having explained the Method of resolving adfected Equations, we proceed to such Questions as produce these Equations.

The

The Manner of solving Questions, when the unknown Quantity has several Powers in one Equation, and only the first Power in the other Equation.

69. WHEN the unknown Quantities are to the first and second Power in one Equation, and but to the first Power in the other Equation, find the Value of that unknown Quantity, in the Equation where its Terms are the more simple; raise this Equation, or Value of the unknown Quantity, to the several Powers of the unknown Quantity in the other Equation; then in that Equation for the several Powers of the unknown Quantity, write, or put these Values, which *exterminates* that unknown Quantity, leaving an Equation with only one unknown Quantity, which may be resolved by some of the Methods already explained.

Question 84. *There are two Numbers, if the Square of the greater is divided by the lesser, to this Quotient adding the greater, from which Sum substracting the Square of the lesser, the Remainder is* 100:

And the Sum of the two Numbers is 50. *What are the Numbers sought?*

Let $a =$ the greater, $e =$ the lesser Number, $m = 100$, $p = 50$.

$$\left.\begin{array}{l|l} 1 & \frac{aa}{e} + a - ee = m \\ 2 & a + e = p \end{array}\right\} \text{By the Question.}$$

In the first Equation both the unknown Quantities are to the first and second Power; but in the second Equation they are only to the first Power; therefore, according to the Directions, find the Value of a or e, in the second Equation.

$$2 - e \mid 3 \mid a = p - e$$

Because a is to the second Power in the first Equation, raise the third Equation to the second Power.

$3 \textcircled{\scriptsize{G}} 2$	4	$aa = pp - 2pe + ee$

Now for aa and a in the firſt Equation, write their reſpective Values, $pp - 2pe + ee$, and $p - e$, found by the third and fourth Equations, then we have,

$1 \cdot 3 \cdot 4$	5	$\frac{pp - 2pe + ee}{e} + p - e - ee = m$, an Equation from which a is exterminated, and contains only the unknown Quantity e.
$5 \times e$	6	$pp - 2pe + ee + pe - ee - eee = me$
That is	7	$pp - pe - eee = me$
$7 + eee$	8	$pp - pe = me + eee$
$8 + pe$	9	$pp = eee + me + pe$
9 in Numbers	10	$eee + 100e + 50e = 2500$
That is	11	$eee + 150e = 2500$

Here the Equation appears to be *adfected,* and to reſolve it, let us ſuppoſe $e = 9$.

Then $eee = 729$
And $150e = 1350$

2079 which being leſs than 2500, therefore e muſt be more than 9.

Then let $r = 9$, and $y =$ what 9 wants of the true Value of e, then by *Caſe* 1, Art. 67, we have,

	1	$r + y = e$
$1 \textcircled{\scriptsize{G}} 3$	2	$rrr + 3rry + 3ryy = eee$, rejecting the Powers of y above yy.

Becauſe in the given Equation e is multiplied by 150, therefore multiply the firſt Step by 150.

$1 \times \overline{150}$	3	$150r + 150y = 150e$

Now add the ſecond and third Equations together, becauſe the like Powers of e in the *adfected* Equation, are connected by the Sign $+$.

$2 + 3$	4	$rrr + 3rry + 3ryy + 150r + 150y = eee + 150e$
But	5	$eee + 150e = 2500$, by the given Equation.

$4 \cdot 5$

4 · 5	6	$rrr + 3rry + 3ryy + 150r + 150y = 2500$
6 in Numbers	7	$729 + 243y + 27yy + 1350 + 150y = 2500$
7 contracted	8	$2079 + 393y + 27yy = 2500$
$8 - \overline{2079}$	9	$393y + 27yy = 421$ Dividing by the Co-efficient of yy.
$9 \div \overline{27}$	10	$14.55y + yy = 15.59$ Now dividing by the Co-efficient of y *plus* y.
$10 \div 14.55 + y$	11	$y = \dfrac{15.59}{14.55 + y}$

Operation 14.55) 15.59 (1. $= y$
1.
———— 15.55
Divisor 15.55 ————
4 Remainder neglected.

$r = 9$
$y = 1$

$r + y = 10 = e$, which being involved and tried will be found to be the true Root: Hence 10 is the lesser Number sought.

Then by the third Step of the Work to the Question $a = p - e = 40$, the greater Number sought.

In the Division for finding y, the Learner may observe, that as the two next Figures in the Quotient will be *Cyphers*, and in the Places of Fractions, and the third Figure being of so small a Value, I proceed no further in the Division, but leave it as in the Work, and so happen to find the true Value of e.

Question 85. *Two Men,* A *and* B, *have such a Number of Pounds, that the Pounds* A *has, divided by the Pounds* B *has, and from this Quotient substracting three times the Square of* B's *Pounds, and to the Remainder adding the Square of* A's *Pounds, the Sum is* 27:

But if from the Pounds A *has, there is substracted the Pounds* B *has, the Remainder is* 5. *How many Pounds had each Man?*

Put $a =$ the Money of A, $e =$ the Money of B, $d = 27$, $x = 5$.

$$\left.\begin{array}{l|l} 1 & \frac{a}{e} - 3ee + aa = d \\ 2 & a - e = x \end{array}\right\}\text{By the Question.}$$

In

In the firſt Equation a and e being to the firſt and ſecond Power, and to the firſt Power only in the ſecond Equation, therefore by the Directions find the Value of a, or e, in the ſecond Equation, ſuppoſe we find the Value of a.

$2+e$	3	$a=x+e$

Raiſe this to the ſecond Power, becauſe a is to the ſecond Power in the firſt Equation.

$3 \circledS 2$	4	$aa=xx+2xe+ee$

Now for a and aa in the firſt Equation, write their reſpective Values, $x+e$, and $xx+2xe+ee$.

$1 \cdot 3 \cdot 4$	5	$\frac{x+e}{e}-3ee+xx+2xe+ee=d$, here a is exterminated, for the Equation contains only the unknown Quantity e.
$5 \times e$	6	$x+e-3eee+xxe+2exe+eee=de$
That is	7	$x+e-2eee+xxe+2xee=de$
7 in Numbers	8	$5+e-2eee+25e+10ee=27e$
$8+2eee$	9	$5+e+25e+10ee=2eee+27e$
$6-10ee$	10	$5+e+25e=2eee-10ee+27e$
$10-e$	11	$5+25e=2eee-10ee+26e$
$11-25e$	12	$2eee-10ee+e=5$

To reſolve this Equation, ſuppoſe $e=6$.

Then $2eee=432$
$-10ee=-360$
$\ 72$
$+e=\ 6$
$\ 78$ which being greater than 5, the Number in the given Equation, hence e cannot be ſo much as 6, therefore,

Let $r=6$, and $y=$ what 6 is too much, then by *Caſe* 2, Page 235.

	1	$r-y=e$
$1 \circledS 3$	2	$rrr-3rry+3ryy=eee$ rejecting the Powers of y above yy.

Becauſe

Becauſe in the given Equation eee is multiplied by 2, therefore multiply the laſt Equation by 2.

$2 \times \overline{2}$	3	$2rrr - 6rry + 6ryy = 2eee$

Now raiſe $r - y = e$ to the ſecond Power, after which multiply it by 10, becauſe it is $10\,ee$ in the given adfected Equation.

$1 \circledS 2$	4	$rr - 2ry + yy = ee$
$4 \times \overline{10}$	5	$10rr - 20ry + 10yy = 10ee$

Then add or ſubſtract the Equations that are equal to $2eee$, $10ee$ and e, according as thoſe Quantities have the Signs + or —, in the given *adfected* Equation.

$3 - 5 + 1$	6	$2rrr - 6rry + 6ryy - 10rr + 20ry - 10yy + r - y = 2eee - 10ee + e$
But	7	$2eee - 10ee + e = 5$, by the given Equation.
6 . 7	8	$2rrr - 6rry + 6ryy - 10rr + 20ry - 10yy + r - y = 5$
8 in Numbers	9	$432 - 216y + 36yy - 360 + 120y - 10yy + 6 - y = 5$
9 contracted	10	$78 - 97y + 26yy = 5$

Tranſpoſe 5 it being leſs than 78

$10 - 5$	11	$73 - 97y + 26yy = 0$, for one Side of the Equation ſubſtracted from the other, the Remainder muſt be *nothing*, both Sides of the Equation being equal to one another.
$11 + 97y$	12	$73 + 26yy = 97y$
$12 - 26yy$	13	$97y - 26yy = 73$ Divide by the Co-efficient of yy.
$13 \div \overline{26}$	14	$3.73y - yy = 2.807$. Now divide by the Co-efficient of y *minus* y.
$14 \div 3.73 - y$	15	$y = \dfrac{2.807}{3.73 - y}$

Operation 3.73) 2.807 (1. $= y$
— 1.
——— 2.73
Diviſor 2.73 ———
7 Remainder neglected.

$r = 6$

$r = 6$ by Suppoſition,

$\underline{-y = -1}$

$r - y = 5 = e$, which being involved and tried it will be found to be the true Root, hence B had 5 Pounds.

Then by the third Step of the Work to the Queſtion $a = x + e = 10$ Pounds, the Money A had.

70. *The Numerical Method of reſolving adfected Equations being explained, we ſhall now ſhow the* Learner, *that every adfected Equation has as many Roots, either* real or imaginary, *as are the higheſt Dimenſions of its unknown Quantity.*

For in any Equation where the higheſt Power of the unknown Quantity is the Biquadratic, or fourth Power, then there may be four Values of the unknown Quantity; if it is only to the third Power, then there may be three Values of the unknown Quantity, and ſo on: But there cannot be more Roots or Values of the unknown Quantity than there are Dimenſions in the Equation.

Theſe Roots are ſometimes *affirmative*, and ſometimes *negative*, and ſome Roots are *impoſſible*. The Reader obſerving how Quadratic Equations were compounded and generated, may better underſtand the Nature of theſe Roots. Thus,

Suppoſe $a = 1$, then $a - 1 = 0$, again ſuppoſe $a = 2$, then $a - 2 = 0$.

Now multiply theſe two together - - $a - 1 = 0$

$$\begin{array}{r} a - 2 = 0 \\ \hline aa - a = 0 \\ -2a + 2 = 0 \\ \hline \end{array}$$

An Equation of two Dimenſions, which has two Roots, *viz.* 1 and 2 $\}$ $aa - 3a + 2 = 0$

Again, let $a = 3$, then - - - $a - 3 = 0$

$$\begin{array}{r} \hline aaa - 3aa + 2a = 0 \\ -3aa + 9a - 6 = 0 \\ \hline \end{array}$$

From multiplying theſe together, we have an Equation of three Dimenſions, and which has 3 Roots, *viz.* 1 . 2 and 3. $\}$ $aaa - 6aa + 11a - 6 = 0$

Laſtly, ſuppoſe $a = -5$, then - - $a + 5 = 0$

$$\begin{array}{r} \hline aaaa - 6aaa + 11aa - 6a = 0 \\ +5aaa - 30aa + 55a - 30 = 0 \\ \hline \end{array}$$

An Equation of 4 Dimenſions, and which has 4 Roots, *viz.* 1 . 2 . 3. and -5, and ſo of any other Power. $\}$ $aaaa - aaa - 19aa + 49a - 30 = 0$

Theſe ſeveral Multiplications muſt all be $= 0$ becauſe the Multiplicand and Multiplier are each $= 0$.

 By

By the same Method that we found the two Roots in Quadratic Equations, we may find the Roots of these Equations. For suppose we had this Equation $aaaa - aaa - 19aa + 49a - 30 = 0$ given, which being resolved by the Method of *Converging Series*, we shall find $a = 1$, whence 1 is one of the Roots of the given adfected Equation; now transpose 1 to make it $a - 1 = 0$, take the given Equation, which being equal to *nothing*, and dividing it by $a - 1$, the Quotient must be equal to *nothing*, thus,

$$
\begin{array}{l}
a-1=0)\ aaaa-aaa-19aa+49a-30=0\ (aaa-19a+30=0 \\
\qquad\quad aaaa-aaa \\
\hline
\qquad\qquad\qquad -19aa+49a-30 \\
\qquad\qquad\qquad -19aa+19a \\
\hline
\qquad\qquad\qquad\qquad\quad 30a-30 \\
\qquad\qquad\qquad\qquad\quad 30a-30 \\
\hline
\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

Here we find the Quotient to be $aaa - 19a + 30 = 0$, and solving this Equation by the Method of *Converging Series*, we shall find $a = 3$, for another of the Roots of the given adfected Equation.

$$
\begin{array}{l}
\text{Then } a-3=0)\ aaa-19a+30=0\ (aa+3a-10=0 \\
\qquad\qquad\qquad aaa-3aa \\
\hline
\qquad\qquad\qquad\qquad 3aa-19a+30 \\
\qquad\qquad\qquad\qquad 3aa-9a \\
\hline
\qquad\qquad\qquad\qquad\quad -10a+30 \\
\qquad\qquad\qquad\qquad\quad -10a+30 \\
\hline
\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Hence we have got this Quadratic Equation $aa + 3a - 10 = 0$, whence $aa + 3a = 10$, the two Roots of which are 2 and -5, the two remaining Roots of the given adfected Equation; in the same Manner all the possible Roots of any other Equation are determined.

And to give the Learner an Instance where some of the Roots of an Equation are impossible:

Suppose $aaa - 4aa + 4a - 16 = 0$, by transposing 16 and resolving the Equation by the Method of *Converging Series*, we shall find $a = 4$: Then transposing 4 to make it $a - 4 = 0$, and making the given Equation equal to *nothing*, and dividing thus,

$$a-4=0)\ aaa-4aa+4a-16=0\ (aa+4=0$$
$$\begin{array}{l} aaa-4aa \\ \hline \qquad\qquad 4a-16 \\ \qquad\qquad 4a-16 \\ \hline \qquad\qquad\quad 0 \end{array}$$

Becaufe the Dividend and Divifor are both equal to *nothing*, therefore the Quotient muft be equal to *nothing*; but if $aa+4=0$, then $aa=-4$ an Equation which has no real or *poffible* Root in Nature, it being impoffible to generate or produce a *negative* Square, for *minus* multiplied into *minus*, as well as *plus* multiplied into *plus*, makes the Product *affirmative*, or *plus*.

Queftion 86. *Three Merchants*, A, B, *and* C, *found the Pounds* A *and* B *had gained, were equal to twice the Pounds* C *had gained:*

But if the Pounds A *gained were added to twice the Pounds* B *gained, and this Sum added to the Pounds* C *gained, it made* 19 *Pounds:*

And the Sum of the Squares of each Perfon's Gain was equal to 77 *Pounds. How much did each Perfon gain?*

Let $a=$ the Gain of A, $e=$ the Gain of B, $y=$ the Gain of C, $m=19$, $p=77$.

	1	$a+e=2y$ By the Queftion.
	2	$a+2e+y=m$
	3	$aa+ee+yy=p$
$1-e$	4	$a=2y-e$. Raife this Equation to the fecond Power, it being aa at the third Equation.
$4 \,\&\, 2$	5	$aa=4yy-4ye+ee$

Now for a, and aa in the fecond and third Equations write their refpective Values, *viz.* $2y-e$, and $4yy-4ye+ee$.

$2 \cdot 4$	6	$3y+e=m$
$3 \cdot 5$	7	$5yy-4ye+2ee=p$.

Here the Queftion is reduced to two Equations and two unknown Quantities, for a is exterminated, therefore in the fixth Equation, find the Value of e, or y, and raife it to the fecond Power, for thofe Quantities are to the fecond Power in the feventh Equation.

$6 - 3y$	8	$e = m - 3y$
$8 \odot 2$	9	$ee = mm - 6my + 9yy$ Multiply this Equation by 2, because it is $2ee$ in the seventh Equation.
9×2	10	$2ee = 2mm - 12my + 18yy$

Now in the seventh Equation for e and $2ee$ write their Values at the eighth and tenth Steps.

7 . 8 . 10	11	$5yy - 4ym + 12yy + 2mm - 12my + 18yy = p$, an Equation with only the unknown Quantity y.
11 contracted	12	$35yy - 16my + 2mm = p$
$12 - 2mm$	13	$35yy - 16my = p - 2mm$, here the Equation appears *quadratic*, and it is likewise *ambiguous*, for $2mm$ is greater than p.
$13 \div 35$	14	$yy - \dfrac{16my}{35} = \dfrac{p - 2mm}{35}$
$14\ c\ \square$	15	$yy - \dfrac{16my}{35} + \dfrac{256mm}{4900} = \dfrac{256mm}{4900} + \dfrac{p - 2mm}{35}$

The Co-efficient of y is $\dfrac{16m}{35}$, which being divided by 2, or $\dfrac{2}{1}$ by the Rule in common Arithmetic for Division of Vulgar Fractions, the Quotient is $\dfrac{16m}{70}$, the Square of which is $\dfrac{256mm}{4900}$.

$15\ \omega\ 2$	16	$y - \dfrac{16m}{70} = \sqrt{\dfrac{256mm}{4900} + \dfrac{p - 2mm}{35}}$
$16 + \dfrac{16m}{70}$	17	$y = \dfrac{16m}{70} \pm \sqrt{\dfrac{256mm}{4900} + \dfrac{p - mm}{35}}$ $= 4.9999$ or 3.6857 But 4.9999 is the Number, the Answer being 5. then if $y = 5$
By the 8th Step	18	$e = m - 3y = 4$
By the 4th Step	19	$a = 2y - e = 6$

Queſtion 87. A, B, *and* C, *having been at the Gaming-Table, found the Pounds* A *loſt added to the Pounds* C *loſt was equal to twice the Pounds* B *loſt :*

But the Pounds A *loſt added to the Pounds* B *loſt, and this added to twice the Pounds* C *loſt, the Sum was* 22 *Pounds :*

And the Product of what A *and* B *loſt, being added to three times the Product of what* B *and* C *loſt, the Sum was* 120 *Pounds. How much did each loſe?*

Let $a =$ the Sum A loſt, $e =$ the Sum B loſt, $y =$ the Sum C loſt, $d = 22$, $n = 120$.

	1	$a + y = 2e$	By the Queſtion.
	2	$a + e + 2y = d$	
	3	$ae + 3ey = n$	
$1 - y$	4	$a = 2e - y$	
$2 \cdot 4$	5	$3e + y = d$	
$3 \cdot 4$	6	$2ee - ey + 3ey = n$	

By the fifth and ſixth Steps, the Queſtion is reduced to two Equations, and two unknown Quantities, and becauſe y is only to the firſt Power in both Equations, find the Value of y in each of them.

$5 - 3e$	7	$y = d - 3e$
$6 - 2ee$	8	$2ey = n - 2ee$
$8 \div 2e$	9	$y = \frac{n - 2ee}{2e}$
$7 \cdot 9$	10	$\frac{n - 2ee}{2e} = d - 3e$
$10 \times 2e$	11	$n - 2ee = 2de - 6ee$
$11 + 6ee$	12	$4ee + n = 2de$
$12 - n$	13	$4ee - 2de = -n$
$13 - 2de$	14	Here the Equation is *quadratic* and *ambiguous.*
$14 \div 4$	15	$ee - \frac{de}{2} = -\frac{n}{4}$
$15\ c\ \square$	16	$ee - \frac{de}{2} + \frac{dd}{16} = \frac{dd}{16} - \frac{n}{4}$ for the Co-efficient of e is $\frac{d}{2}$, which being divided by 2 as in the laſt Queſtion, the

		the Quotient is $\frac{d}{4}$, the Square of which is $\frac{dd}{16}$.
16 ɯ 2	17	$e - \frac{d}{4} = \sqrt{\frac{dd}{16} - \frac{n}{4}}$
$17 + \frac{d}{4}$	18	$e = \frac{d}{4} \pm \sqrt{\frac{dd}{16} - \frac{n}{4}} = 5\cdot5 \pm \cdot5$ ($= 6$, or 5, if $e = 6$,
Then by Step 7th	19	$y = d - 3e = 4$
And by Step 4th	20	$a = 2e - y = 8$

But if $e = 5$, then by the seventh Step $y = d - 3e = 7$, and by the fourth Step $a = 2e - y = 3$.

Question 88. *There are two Numbers, the Sum of their Squares being added to their Sum, is* 338:
And their Product is 156. *What are the Numbers?*

Let a and e be the two Numbers sought, $b = 338$, $m = 156$.

Then	1	$aa + ee + a + e = b$ } By the Question.
	2	$ae = m$
$2 \div e$	3	$a = \frac{m}{e}$
3 ⊛ 2	4	$aa = \frac{mm}{ee}$
1 · 3 · 4	5	$\frac{mm}{ee} + ee + \frac{m}{e} + e = b$, an Equation having the unknown Quantity e only.
$5 \times ee$	6	$mm + eeee + \frac{eem}{e} + eee = bee$
		But as $\frac{eem}{e} = em$, the e being rejected (by Art. 20.
Hence	7	$mm + eeee + em + eee = bee$

There being only the known Quantity mm, transpose the others so that mm may be at last *affirmative*; and this is to be observed,

observed, that in transposing the Quantities in these *adfected* Equations, the Side of the Equation which is known may at last be *affirmative*.

7 — $eeee$	8	$mm + em + eee = bee - eeee$
8 — eee	9	$mm + em = bee - eeee - eee$
9 — em	10	$mm = bee - eeee - eee - me$
Or	11	$-eeee - eee + bee - me = mm$, it being the common Method to place these Equations, according to the highest Power of the unknown Quantity.
11 in Numbers	12	$-eeee - eee + 338ee - 156e = 24336$.

Now suppose $e = 10$.

$$\begin{aligned} \text{Then } -eeee &= -10000 \\ -eee &= -1000 \\ &\overline{-11000} \\ +338ee &= 33800 \\ &\overline{22800} \\ -156e &= -1560 \end{aligned}$$

$-eeee - eee + 338ee - 156e = 21240$ which being less than 24336 the Number in the given Equation, therefore e must be more than 10.

Let $r = 10$, and put $y =$ what it wants of being the true Root.

Then	1	$r + y = e$
1 ⊕ 4	2	$rrrr + 4rrry + 6rryy = eeee$, all the Powers of y above yy being rejected.
1 ⊕ 3	3	$rrr + 3rry + 3ryy = eee$, rejecting all the Powers of y above yy.
1 ⊕ 2	4	$rr + 2ry + yy = ee$

Because in the given Equation it is $338ee$, therefore multiply the fourth Equation by 338.

$4 \times \overline{338}$	5	$338rr + 676ry + 338yy = 338ee$

Because

Because in the given Equation it is $156\,e$, therefore multiply the first Equation by 156.

1 $\overline{156}$	6	$156r+156y=156e$

Now the second, third, fifth, and sixth Equations being equal to the several Powers of e, and multiplied by the same Co-efficients as in the given Equation, add or substract them according to the Signs those Powers have in that Equation.

$-2-3+5-6$	7	$-rrrr-4rrry-6rryy-rrr-3rry$ $-3ryy+338rr+676ry+338yy$ $-156r-156y=-eeee-eee$ $+338ee-156e$
But	8	$-eeee-eee+338ee-156e=24336$ by the given Equation.
7 . 8	9	$-rrrr-4rrry-6rryy-rrr-3rry$ $-3ryy+338rr+676ry+338yy$ $-156r-156y=24336$
9 in Numbers	10	$-10000-4000y-600yy-1000$ $-300y-30yy+33800+6760y$ $+338yy-1560-156y=24336$
10 contracted	11	$21240+2304y-292yy=24336$
11 — 21240	12	$2304y-292yy=3096$ Now divide by the Co-efficient of yy.
12 $\div\,\overline{292}$	13	$7.89y-yy=10.6$ And dividing by the Co-efficient of y *minus y*,
13 $\div$ 7.89 — y	14	$y=\dfrac{10.6}{7.89-y}$

Operation 7.89) 10.60 (1.7 = y
— 1.
——— 6.89
Divisor 6.89 ———
— 1.7   3710
———   3633
Divisor 5.19 ———
77

$r=10$ by Supposition.
$+y=\underline{\ 1.7\ }$
$r+y=11.7$ which being involved and tried, it will be found too little; therefore for a second Operation,

Suppose $r=11.7$ and y what it wants of the true Root.

Then

Then	1	$r+y=e$
1 ⊙ 4	2	$rrrr+4rrry+6rryy=eeee$, the Powers of y above yy being rejected.
1 ⊙ 3	3	$rrr+3rry+3ryy=eee$, the Powers of y above yy being rejected.
1 ⊙ 2	4	$rr+2ry+yy=ee$

Becauſe in the given Equation it is 338 ee, therefore multiply the laſt Equation by 338.

$4\times\overline{338}$	5	$338rr+676ry+338yy=338ee$

Becauſe in the given Equation it is 156 e, therefore multiply the firſt Equation by 156.

$1\times\overline{156}$	6	$156r+156y=156e$

Now add or ſubſtract the Equations that are equal to $eeee$, eee, 338 ee and 156 e, according to the Signs thoſe Quantities have in the given *adfected* Equation.

$-2-3+5-6$	7	$-rrrr-4rrry-6rryy-rrr-3rry-3ryy+338rr+676ry+338yy-156r-156y=-eeee-eee+338ee-156e$
But	8	$-eeee-eee+338ee-156e=24336$, by the given Equation.
7 . 8	9	$-rrrr-4rrry-6rryy-rrr-3rry-3ryy+338rr+676ry+338yy-156r-156y=24336$
9 in Numbers	10	$-18738.8721-6406.452y-821.34yy-1601.613-410.67y-35.1yy+46268.82+7909.2y+338yy-1825.2-156y=24336$
10 contracted	11	$24103.1349+936.078y-518.44yy=24336$
11 —	12	$936.078y-518.44yy=232.8651$ Dividing by the Co-efficient of yy.
12 ÷	13	$1.805y-yy=.4491$. Now dividing by the Co-efficient of y *minus* y, that is, by $1.805-y$.
13 ÷ 1.805 — y	14	$y=\dfrac{4491}{1.805-y}$

Operation

Operation 1.805) .4491 (.297 $= y$.
— .2

	— .2	
Divisor	1.605	3210
	— .29	
		12810
Divisor	1.315	11835
	— 97	
		9750
Divisor	1.218	8526
		1224

$r = 11.7$ by Supposition,
$+y = \ \ .297$

$r + y$ is $11.997 = e$, which is something too little, the true Value being 12. but this may inform the Learner of the Nature of solving these high *adfected* Equations, every Operation approaching nearer and nearer to the true Root, from whence it may be found to any assignable Degree of Exactness.

And having found e to be 12, then by the third Step of the Work to the Question, we have $a = \frac{m}{e} = 13$, the other Number sought.

71. *The Method of resolving Equations when the unknown Quantity is to several Powers in both Equations.*

When both the unknown Quantities are to the first and second Power in both Equations, find the Value of the Square of the unknown Quantity in each Equation, and make these two Equations equal to one another; which Equation will have the first Power only of the unknown Quantity, its Square being exterminated by that Equation.

Then find the Value of the first Power of the unknown Quantity in this last Equation, which raise to the second Power; and in either of the two given Equations in which it may be most conveniently done, for this unknown Quantity and its several Powers, write their respective Values, which will give an Equation with only one unknown Quantity, and is to be reduced by the Rules already explained.

Question 89. *To find two Numbers, the Sum of whose Squares is equal to the lesser multiplied by* 20:

And the Square of the lesser being added to their Product, the Sum is 16.

Let $a =$ the greater Number, $e =$ the lesser Number, $m = 20$, $d = 16$.

	1	$aa + ee = me$ } By the Question.
	2	$ee + ae = d$ }
		Begin to exterminate ee according to the Directions, that is, find the Value of ee in both the given Equations.
$1 - aa$	3	$ee = me - aa$
$2 - ae$	4	$ee = d - ae$
$3 \cdot 4$	5	$me - aa = d - ae$, here ee is exterminated, now find the Value of e.
$5 + ae$	6	$me + ae - aa = d$
$6 + aa$	7	$me + ae = d + aa$
$7 \div m + a$	8	$e = \dfrac{d + aa}{m + a}$
		Raise this Value of e to the second Power.
8 ⊕ 2	9	$ee = \dfrac{dd + 2daa + aaaa}{mm + 2ma + aa}$

Now in the first Equation for ee and e, write their respective Values at the eighth and ninth Steps.

$1 \cdot 9 \cdot 8$	10	$aa + \dfrac{dd + 2daa + aaaa}{mm + 2ma + aa} = \dfrac{md + maa}{m + a}$ an Equation clear of e, having only the unknown Quantity a.

To clear this Equation of the Fractions, observe that $mm + 2ma + aa$ is the Square of $m + a$, the former arising from the Involution of the latter by the eighth and ninth Steps, and in the Multiplication of Fractions, it being the same thing to divide the Divisor, as to multiply the Dividend, to multiply $\dfrac{dd + 2daa + aaaa}{mm + 2ma + aa}$, by $m + a$, we only change the Divisor to $m + a$, that being the Quotient of $mm + 2ma + aa$ divided by $m + a$, the rest of the Multiplication is the same as usual.

$10 \times m+a$	11	$maa+aaa+\frac{dd+2daa+aaaa}{m+a}$ $=md+maa$
$11 \times m+a$	12	$mmaa+maaa+maaa+aaaa+dd$ $+2daa+aaaa=mmd+mmaa$ $+mda+maaa$
$12-maaa$	13	$mmaa+maaa+aaaa+dd+2daa$ $+aaaa=mmd+mmaa+mda$
$14-mmaa$	14	$maaa+aaaa+dd+2daa+aaaa$ $=mmd+mda$
$14-mda$	15	$maaa+2aaaa+dd+2daa-$ $mda=mmd$
$15-dd$	16	$2aaaa+maaa+2daa-mda$ $=mmd-dd$
16 in Numbers	17	$2aaaa+20aaa+32aa-320a=6144$ Because the Co-efficient of $aaaa$ will divide the other Co-efficients without any Remainder, divide by it.
$17 \div 2$	18	$aaaa+10aaa+16aa-160a=3072$

Which Equation being resolved by the Method of *Converging Series*, we shall find $a=6$, or nearly to it, 6 being the true Root, from whence by the eighth Step $e=2$.

Question 90. *There are two Numbers, if the greater is added to its Square, and from this Sum we substract the Square of the lesser, the Remainder is* 94:

But the Square of the lesser, being added to the lesser, this Sum is equal to twice the greater.

Let $a=$ the greater Number, $e=$ the lesser Number, $m=94$.

	1	$a+aa-ee=m$ } By the Question.
	2	$ee+e=2a$ }
		Begin with finding the Value of ee in each Equation.
$1+ee$	3	$aa+a=m+ee$
$3-m$	4	$aa+a-m=ee$
$2-e$	5	$ee=2a-e$
$4 \cdot 5$	6	$aa+a-m=2a-e$, here ee is exterminated, now find the Value of e.
$6+e$	7	$e+aa+a-m=2a$
$7-a$	8	$e+aa-m=a$

$8+m$	9	$e+aa=a+m$
$9-aa$	10	$e=a+m-aa$
		Raiſe this Value of e to the ſecond Power.
10 ☉ 2	11	$ee=aa+2am+mm-2aaa-2maa+aaaa$

Now for $e\,e$ and e in the ſecond Equation, write their reſpective Values, found at the tenth and eleventh Steps.

2 . 11 . 10	12	$aa+2am+mm-2aaa-2maa+aaaa+a+m-aa=2a$
12 in Numbers	13	$188a+8836-2aaa-188aa+aaaa+a+94=2a$
13 contracted	14	$187a+8930-2aaa-188aa+aaaa=0$
		Tranſpoſe the ſeveral Powers of a, that 8930 the known Part of the Equation may be *affirmative.*
$14-aaaa$	15	$-aaaa=187a+8930-2aaa-188aa$
$15+2aaa$	16	$-aaaa+2aaa=187a+8930-188aa$
$16+188aa$	17	$-aaaa+2aaa+188aa=187a+8930$
$17-187a$	18	$-aaaa+2aaa+188aa-187a=8930$

Which Equation being reſolved, we ſhall find $a=10$, or nearly to it, 10 being the true Root.

Then by the tenth Step $e=a+m-aa=4$.

We ſhall now proceed to the Solution of ſeveral Geometrical Problems upon the ſame *general* Principles, and if the Learner is not ſufficiently acquainted with the Elements of Geometry, to diſcover how the Equations are formed from the Properties of the Figure, he may omit theſe Queſtions, and proceed to the others which require no Knowledge in Geometry.

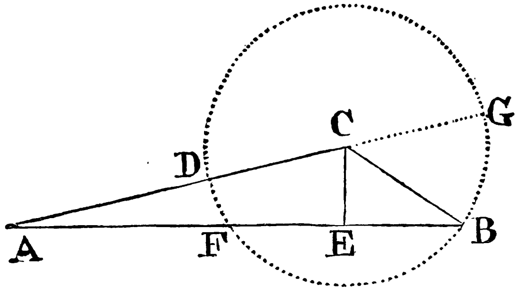
Queſtion 91. *In the oblique Triangle* ABC, *given the Difference between the Sides* AC *and* BC $=8$, *and the Difference between the Segments of the Baſe* AE *and* EB $=10$, *and the Perpendicular* CE, *let fall from the vertical Angle* C, *upon the Baſe* AB $=16$. *To find the Sides* AC, CB, *and Baſe* AB?

Upon C as a Center, with the Radius CB, draw the Circle GBFD, and continue AC to G.

Hence CB $=$ CD, as Radius of the ſame Circle, whence AD is the Difference of the Sides, or the Difference between AC and CB $=8$.

And

And $BE = EF$, as FB is bisected at E by the $3 . e . 3$ hence AF is the Difference of the Segments of the Base, or the Difference between AE and $EB = 10$.



Let $AD = d = 8$, and $DC = CB = e$, then $AC = d + e$.

Let $AF = b = 10$, and $FE = EB = a$, hence $AB = b + 2a$.

Let $CE = p = 16$.

Having two unknown Quantities, a and e, and no Equation from the Conditions of the Queston, we must raise two Equations from the Properties of the Figure.

Now the Lines AG and AB are drawn from within a Circle, and touch without at the Point A, therefore by $37 . e . 3$ $AG \times AD = AB \times AF$, all which Lines are expressed in Symbols, except AG, but $CG = CD = e$, and $AC = d + e$, therefore $AG = d + 2e$, hence we have in Symbols,

	1	$\overline{d + 2e} \times d = \overline{b + 2a} \times b$ the short Lines over the Quantities, signifies that they are both to be multiplied by the Quantity which follows the Sign of Multiplication.
		But the Triangle CEB is right-angled, therefore by
$47 . e . 1$	2	$pp + aa = ee$
From the first	3	$dd + 2de = bb + 2ba$
		Now find the Value of either a or e, in the third Equation.
$3 - dd$	4	$2de = bb + 2ba - dd$

But as we shall have Occasion to square this Equation, for when the Value of e is found, that Equation must be raised to the second

fecond Power, it being ee in the fecond Equation; and $bb - dd$ being a known Quantity, to avoid Trouble, fubftitute $x = bb - dd$.

Then	5	$2de = x + 2ba$
$5 \div 2d$	6	$e = \frac{x + 2ba}{2d}$
		Raife this to the fecond Power, becaufe it is ee in the fecond Equation.
6 ⊝ 2	7	$ee = \frac{xx + 4xba + 4bbaa}{4dd}$
2 . 7	8	$\frac{xx + 4xba + 4bbaa}{4dd} = pp + aa$, (an Equation clear of e.
$8 \times 4dd$	9	$xx + 4xba + 4bbaa = 4ddpp + 4ddaa$
		Bring all the Powers of a to one Side of the Equation.
$9 - 4ddaa$	10	$4bbaa - 4ddaa + 4xba + xx = 4ddpp$
$10 - xx$	11	$4bbaa - 4ddaa + 4xba = 4ddpp - xx$

Here the Equation appears *quadratic*, the unknown Quantity being only to the firft and fecond Power; but as the Square of the unknown Quantity has Co-efficients, therefore by Article 58, divide the Equation by $4bb - 4dd$, the Co-efficients of aa.

$11 \div 4bb - 4dd$	12	$aa + \frac{4xba}{4bb - 4dd} = \frac{4ddpp - xx}{4bb - 4dd}$

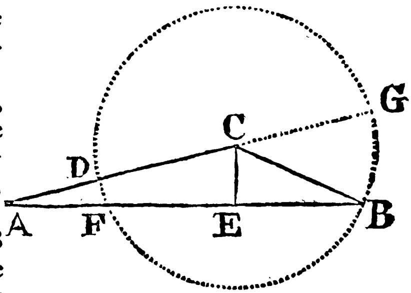
The Work being now prepared for *compleating the Square*, and the *Co-efficient* of a being a Fraction, to fave the Trouble of dividing it by 2, and fquaring the Quotient according to Art. 57, fubftitute $y = \frac{4xb}{4bb - 4dd}$,

Then	13	$aa + ya = \frac{4ddpp - xx}{4bb - 4dd}$
13 *c* □	14	$aa + ya + \frac{yy}{4} = \frac{4ddpp - xx}{4bb - 4dd} + \frac{yy}{4}$
14 ɯ 2	15	$a + \frac{y}{2} = \sqrt{\frac{4ddpp - xx}{4bb - 4dd} + \frac{yy}{4}}$
$15 - \frac{y}{2}$	16	$a = \sqrt{\frac{4ddpp - xx}{4bb - 4dd} + \frac{yy}{4}} : - \frac{y}{2}$ $= 16.7.$

Then

Then by Step 6th	17	$e = \dfrac{x + 2ba}{2d} = 23.12$
Hence	18	$AC = d + e = 31.12$
	19	$CB = e = 23.12$
	20	$AB = b + 2a = 43.4$

Queſtion 92. *In the oblique Triangle* ABC, *there is given the Sum of the Sides* AC *and* BC = 8, *and the Difference of the Segments of the Baſe* AE *and* BE = 2, *with the Perpendicular* CE, *let fall from the vertical Angle at* C *upon the Baſe* AB = 1. *To find the Sides* AC, BC, *and Baſe* AB?



Upon C as a Center with the Radius CB, draw the Circle GBFD, and continue AC to G.

Then CG = CB = CD, being all Radii of the ſame Circle, whence AG is the Sum of the Sides, or AC + CB = 8.

And FE = EB, for FB is biſected at E, by the 3 . *e* . 3, whence AF is the Difference of the Segments of the Baſe, or the Difference between AE and BE = 2.

The Conſtruction of this Figure being the ſame as the laſt, we can raiſe the ſame two Equations from the Figure, but inſtead of AD being given, we have AG given. Let AG, or AC + CB = s = 8, and DC = CG = a, whence DG = 2a, then AG — DG = AD = $s - 2a$.

Put AF = d = 2, and FE = EB = e, then AB = $d + 2e$, let CE = p = 1.

Now as in the laſt Queſtion, becauſe the Lines AG and AB are drawn from the Circumference within the Circle, and touch at the Point A without the Circle, hence by 37 . *e* . 3 AG × AD = AB × AF, that is,

in Symbols	1	$s \times \overline{s - 2a} = \overline{d + 2e} \times d$
That is	2	$ss - 2sa = dd + 2de$
by 47 . *e* . 1	3	$pp + ee = aa$ the Triangle CEB being right-angled, and DC = CB = a.

Here the Question is expreſſed by two Equations, and two unknown Quantities.

$2 + 2sa$	4	$ss = dd + 2de + 2sa$
$4 - dd$	5	$ss - dd = 2de + 2sa$
$5 - 2de$	6	$2sa = ss - dd - 2de$
		Subſtitute as before $x = ss - dd$, for when the Value of a is found, it muſt be raiſed to the ſecond Power, to compare it with aa in the third Equation.
Then	7	$2sa = x - 2de$
$7 \div 2s$	8	$a = \dfrac{x - 2de}{2s}$
		Raiſe this Equation to the ſecond Power, becauſe it is aa in the third Equation.
$8 \ominus 2$	9	$aa = \dfrac{xx - 4xde + 4ddee}{4ss}$
$3 \cdot 9$	10	$pp + ee = \dfrac{xx - 4xde + 4ddee}{4ss}$
$10 \times 4ss$	11	$4sspp + 4ssee = xx - 4xde + 4ddee$
		Now bring all the Powers of e to one Side of the Equation.
$11 - 4ddee$	12	$4ssee - 4ddee + 4sspp = xx - 4xde$
$12 + 4xde$	13	$4ssee - 4ddee + 4xde + 4sspp = xx$
$13 - 4sspp$	14	$4ssee - 4ddee + 4xde = xx - 4sspp$

Here the Equation appears *quadratic*, but is not *ambiguous*, for xx is greater than $4sspp$. Dividing by the Co-efficient of ee, by Art. 58.

$14 \div 4ss - 4dd$	15	$ee + \dfrac{4xde}{4ss - 4dd} = \dfrac{xx - 4sspp}{4ss - 4dd}$

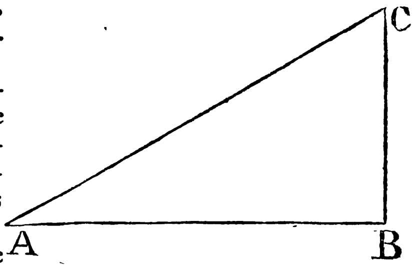
The Work being prepared for *compleating the Square*, ſubſtitute $y = \dfrac{4xd}{4ss - 4dd} = \dfrac{xd}{ss - dd}$,

Then	16	$ee + ye = \dfrac{xx - 4sspp}{4ss - 4dd}$
$16\ c\ \square$	17	$ee + ye + \dfrac{yy}{4} = \dfrac{xx - 4sspp}{4ss - 4dd} + \dfrac{yy}{4}$

17 ⧜ 2	18	$e+\frac{y}{2}=\sqrt{\frac{xx-4sspp}{4ss-4dd}+\frac{yy}{4}}$
$18-\frac{y}{2}$	19	$e=\sqrt{\frac{xx-4sspp}{4ss-4dd}+\frac{yy}{4}}:-\frac{y}{2}$ $=2.86$
Then by Step 8th	20	$a=\frac{x-2de}{2s}=3.03=BC.$
Hence	21	$AC=s-a=4.97$
And	22	$BA=d+2e=7.72$

Question 93. *In the right-angled Triangle* ABC, *there is given the Area of the Triangle equal to* 24, *and the Sum of the Hypothenuse* AC *and Perpendicular* BC *equal to* 16. *To find the Sides of the Triangle?*

Let $AC=y$, $AB=a$, $BC=e$, $s=24$, $d=16$.



Here being three unknown Quantities, there must be raised three Equations from the Question, and the Properties of the Figure.

Now as the Triangle ABC is right-angled, therefore,

By 47 . *e* . 1	1	$aa+ee=yy$ } By the Question.
	2	$y+e=d$
	3	$\frac{ae}{2}=s$, from the Rule for finding the Area of the Triangle, for the

Product of the Base and Perpendicular of any Triangle being divided by 2, the Quotient is the Area.

The first Equation has all the three unknown Quantities, but the other two have only two of them. Now if we take that Quantity which is in all the three Equations, and find the Value of it in one of them, and in its Room write that Value in the other two Equations, the Question will be then reduced to two Equations, and two unknown Quantities; thus in the second Equation find the Value of *e*.

$2 - y$	4	$e = d - y$ But as it is ee in the firſt Equation, therefore,
$4 \odot 2$	5	$ee = dd - 2dy + yy$
$1 \cdot 5$	6	$aa + dd - 2dy + yy = yy$ } Two Equations with only two unknown Quantities.
$3 \cdot 4$	7	$\frac{ad - ay}{2} = s$
		Find the Value of y, in each of theſe Equations.
$6 - yy$	8	$aa + dd - 2dy = 0$, for yy being taken away by the Subſtraction, that Side of the Equation is *nothing*.
$9 + 2dy$	9	$2dy = aa + dd$
$9 \div 2d$	10	$y = \frac{aa + dd}{2d}$
7×2	11	$ad - ay = 2s$
$11 + ay$	12	$ad = 2s + ay$
$12 - 2s$	13	$ay = ad - 2s$
$13 \div a$	14	$y = \frac{ad - 2s}{a}$
$10 \cdot 14$	15	$\frac{aa + dd}{2d} = \frac{ad - 2s}{a}$
$15 \times a$	16	$\frac{aaa + dda}{2d} = ad - 2s$
$16 \times 2d$	17	$aaa + dda = 2dda - 4ds$
17 in Numbers	18	$aaa + 256a = 512a - 1536$
$18 - 512a$	19	$aaa - 256a = -1536$

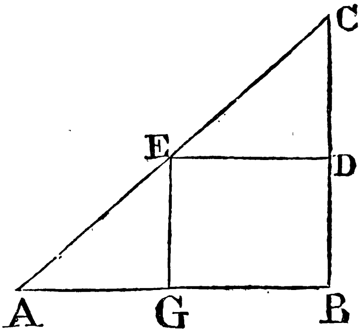
Here the Equation is *adfected*, therefore tranſpoſe the Quantity ſo that the Side of the Equation which is known may have the *affirmative* Sign.

$19 + 256a$	20	$aaa = 256a - 1536$
$20 + 1536$	21	$aaa + 1536 = 256a$
$21 - aaa$	22	$-aaa + 256a = 1536$

Which Equation being reſolved by the Method of *Converging Series*, we ſhall find $a = 8$ nearly, for 8 is the true Root; from whence the other Sides of the Triangle are eaſily determined.

Queſtion 94. *In the right-angled Triangle* ABC, *there is drawn* GE *parallel to the Perpendicular* BC, *given the Perpendicular* BC = 24, *the Segment of the Hypothenuſe* EC = 15, *and*

and the Segment of the Base A G = 20. *To find the Hypothenuse* A C *and Base* A B?



Draw E D parallel to A B.

Let $BC = c = 24$, $EC = n = 15$, $AG = b = 20$, $GB = ED = a$, then $AB = b + a$, $AE = e$, then $AC = n + e$.

Here being two unknown Quantities, we must raise two Equations.

Now the Triangles A G E and E D C are similar, hence,

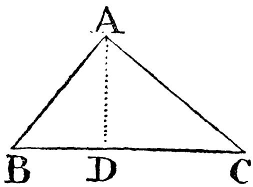
by 4 . *e*, 6	1	$AG : AE :: ED : EC$
in Symbols	2	$b : e :: a : n$
whence	3	$ae = bn$, for Quantities that are in continual Proportion, the Product of the Extreams and Means are equal.
by 47 . *e* . 1	4	$bb + 2ba + aa + cc = nn + 2ne + ee$, the Triangle ABC being right-angled, and as these two last Equations contain the Question, therefore
$3 \div e$	5	$a = \frac{bn}{e}$
5 ⊕ 2	6	$aa = \frac{bbnn}{ee}$
4 . 5 . 6	7	$bb + \frac{2bbn}{e} + \frac{bbnn}{ee} + cc = nn + 2ne + ee$
$7 \times e$	8	$bbe + 2bbn + \frac{bbnn}{e} + cce = nne + 2nee + eee$
$8 \times e$	9	$bbee + 2bbne + bbnn + ccee = nnee + 2neee + eeee$
9 in Numbers	10	$400ee + 12000e + 90000 + 576ee = 225ee + 30eee + eeee$
that is	11	$976ee + 12000e + 90000 = 225ee + 30eee + eeee$
$11 - 225ee$	12	$751ee + 12000e + 90000 = 30eee + eeee$
$12 - 751ee$	13	$eeee + 30eee - 751ee = 12000e + 90000$
$13 - 12000e$	14	$eeee + 30eee - 751ee - 12000e = 90000$

 Which

Which Equation being refolved by the Method of *Converging Series*, we fhall find $e = 25$ nearly, for 25 is the true Root.

Then by the fifth Step $a = 12$, whence $AC = n + e = 40$, and $AB = b + a = 32$.

Queftion 95. *In the Triangle* ABC, *given the Bafe* BC $= 42.5$ *and the Angle at* B $= 49° : 45'$ *and the Angle at* C $= 42° : 30'$ *to find the Perpendicular* AD, *let fall from the vertical Angle at* A *upon the Bafe* BC.



The Triangle ADB is right-angled, and the Angle ABD being given, all the Angles of the Triangle ABD are known, therefore by plain Trigonometry, we can find the Ratio between the Sides BD, and AD, though we do not know the Length of either of them, for as the Sine of the Angle BAD, is to the Logarithm of any Number affumed for the Side BD, fo is the Sine of the Angle at B to a fourth Number, which is the Logarithm of the proportional Number for the Side AD.

Therefore affuming *Unity*, or 1, for the Side BD, we have

As the Sine of the Angle at A	-	40° : 15′	-	9.810316
Is to the Log. of the Side BD	-	1.	- -	0.0
So is the Sine of the Angle at B	-	49° : 45′	-	9.882657
				9.882657
				9.810316
To the Log. of the Side AD	-	1.18	- -	.072341

Hence the Sides AD and BD are to one another, as 1.18 is to 1.

Now let $BC = b = 42.5$ $AD = a$, $m = 1.18$ and $p = 1$. Confequently,

$$1 \mid m : p :: a : \frac{pa}{m} = BD,$$ that is, as the Numbers which exprefs the Proportion of AD and DB, are to one another, fo is the true Length of AD to the true Length of BD.

By the fame Reafoning, in the Triangle ADC, becaufe all the Angles are known, therefore the Ratio of the Sides AD and DC are known, and affuming AD to be $= 1$. and proceeding by Trigonometry as before, we fhall find the proportional Number for CD to be 1.1

Now

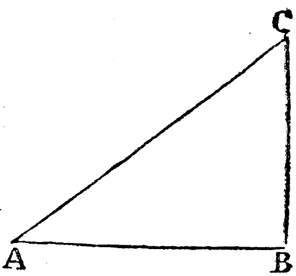
Now putting $d = 1.1$ and $p = 1$, as before, we have

	2	$p : d :: a : \frac{da}{p} =$ DC, by the same Reasoning as at the first Step.

And as we have now expressed in Symbols the two Parts BD and DC of the Base BC,

Hence	3	$\frac{pa}{m} + \frac{da}{p} = b$, that is, BD+DC=BC.
$3 \times m$	4	$pa + \frac{mda}{p} = mb$
$4 \times p$	5	$ppa + mda = mbp$
$5 \div pp + mb$	6	$a = \frac{mbp}{pp + md} = 21.82 =$ AD.

Question 96. *In the right-angled Triangle* ABC, *there is given the Sum of the Sides equal to* 12, *and the Area equal to* 6. *To find the Hypothenuse* AC?



Let $s = 12$, $b = 6$, BC $= a$, AC $= y$.

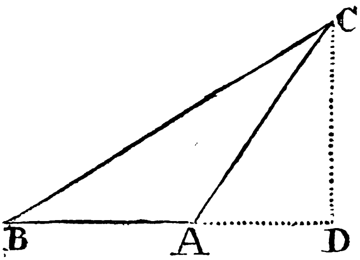
Then by the Property of the Triangle, AB $= \sqrt{yy - aa}$.

Hence	1	$a + y + \sqrt{yy - aa} = s$ by the (Question.
	2	$\frac{a}{2}\sqrt{yy - aa} = b$, from the Rule for (finding the Area of the Triangle.

Now because there is the same Surd in both Equations, find what the Surd is equal to in the second Equation, and write its Value for the Surd in the first Equation.

$2 \div \frac{a}{2}$	3	$\sqrt{yy - aa} = \frac{2b}{a}$ for $b = \frac{b}{1}$ and $\frac{b}{1} \div \frac{a}{2} = \frac{2b}{a}$, by the Rule for Division of Fractions in common Arithmetic.

$1 \cdot 3$	4	$a + y + \frac{2b}{a} = s$
$4 \times a$	5	$aa + ay + 2b = sa$, now the Queſtion is contained in the firſt and fifth Equations.
$1 -$	6	$\sqrt{yy - aa} = s - a - y$
$6 \odot 2$	7	$yy - aa = ss - 2sa - 2sy + 2ay + aa + yy$
$7 \pm$	8	$2aa = 2sa + 2sy - 2ay - ss$
$5 -$	9	$aa = sa - ay - 2b$
9×2	10	$2aa = 2sa - 2ay - 4b$
$8 \cdot 10$	11	$2sa - 2ay - 4b = 2sa + 2sy - 2ay - ss$
$11 + 2ay$	12	$2sa - 4b = 2sa + 2sy - ss$
$12 - 2sa$	13	$-4b = 2sy - ss$
$13 + ss$	14	$2sy = ss - 4b$
$13 \div 2s$	15	$y = \frac{ss - 4b}{2s} = 5 = AC.$



Queſtion 97. *In the Triangle* A B C, *there is given the Sum of the Sides* $BC + BA + AC = 85$, *the Area* $= 200$, *and the Angle at* $A = 124°$. *To find the Sides of the Triangle?*

Let $s = 85$, $b = 200$, $AC = a$, becauſe the Angle $BAC = 124°$, the Angle $CAD = 56°$, and C D being a Perpendicular let fall on A B continued, all the Angles of the Triangle A C D are known, conſequently the Ratio between A C and C D is known, for aſſuming C D to be *Unity*, or 1, then in the Triangle A C D by Trigonometry,

As the Sine of the Angle C A D	-	56° : 00′	- 9.918574
Is to the Log. of the Side C D	-	1.	- 0.000000
So is the Radius - - -	-	90° : 00′	- 10.000000
			10.000000
			9.918574
To the Log. of the Side A C	-	1.21	- 0.081426

Hence we know the Sides A C and C D are as 1.21 to 1.

Calling $m = 1.21$ $d = 1$, therefore,

1	$m : d :: a : \frac{da}{m} = CD$, as at the firſt or ſecond Steps, Queſtion 95.

Now B A being conſidered as the Baſe of the Triangle B A C, and C D as its Perpendicular, hence by the Rule for finding the Area of the Triangle, $BA \times DC = 2b$, that is in

Symbols	2	$2b = \frac{da}{m} \times BA$.
$2 \div \frac{da}{m}$	3	$\frac{2bm}{da} = BA$, for $2b$ or $\frac{2b}{1} \div \frac{da}{m} = \frac{2bm}{da}$ by the Rule for Diviſion of Vulgar Fractions.
And	4	$AD = \sqrt{aa - \frac{ddaa}{mm}}$ for the Triangle A C D is right-angled, where $a = AC$, and $\frac{da}{m} = CD$.
$3 + 4$	5	$\frac{2bm}{da} + \sqrt{aa - \frac{ddaa}{mm}} = BA + AD = BD$
$5 \odot 2$	6	$\frac{4bbmm}{ddaa} + \frac{4bm}{da}\sqrt{aa - \frac{ddaa}{mm}} + aa - \frac{ddaa}{mm} = \overline{BD}^2$, or B D ſquared
$1 \odot 2$	7	$\frac{ddaa}{mm} = \overline{CD}^2$, or C D ſquared.
$6 + 7$	8	$\frac{4bbmm}{ddaa} + \frac{4bm}{da}\sqrt{aa - \frac{ddaa}{mm}} + aa = \overline{CB}^2$, or CB ſquared.

Having now got an Expreſſion equal to the Square of C B, we muſt endeavour to find another Expreſſion for C B from ſome other *Data*.

Now the Sum of all the Sides is given, that is,

	9	$BC + AC + AB = s$
But	10	$AC = a$, and $AB = \frac{2bm}{da}$ by the third Step.
9 . 10	11	$BC + a + \frac{2bm}{da} = s$
$11 - \frac{2bm}{da}$	12	$s - \frac{2bm}{da} = BC + a$
$12 - a$	13	$s - \frac{2bm}{da} - a = BC$
13 ⊕ 2	14	$ss - \frac{4sbm}{da} - 2sa + \frac{4bma}{da} + \frac{4bbmm}{ddaa} + aa = \overline{BC}^{2}$, or BC squared.
8 . 14	15	$ss - \frac{4sbm}{da} - 2sa + \frac{4bma}{da} + \frac{4bbmm}{ddaa} + aa = \frac{4bbmm}{ddaa} + \frac{4bm}{da}\sqrt{aa - \frac{ddaa}{mm}} + aa$
$15 - \frac{4bbmm}{ddaa}$	16	$ss - \frac{4sbm}{da} - 2sa + \frac{4bma}{da} + aa = \frac{4bm}{da}\sqrt{aa - \frac{ddaa}{mm}} : + aa$
$16 - aa$	17	$ss - \frac{4sbm}{da} - 2sa + \frac{4bma}{da} = \frac{4bm}{da}\sqrt{aa - \frac{ddaa}{mm}}$

Here the Learner may obſerve that the unknown Quantity is under the radical Sign, and therefore as ſuch Equations are generally ſquared to take away the Surd, the ſame is to be done here; but as it is aa in all the Quantities under the radical Sign, we can extract the ſquare Root of aa, and join it to the rational Quantity, leaving the remaining Part of the *Surd* under the radical Sign, thus $\frac{4bm}{da}\sqrt{aa - \frac{ddaa}{mm}} = \frac{4bm}{da}a\sqrt{1 - \frac{dd}{mm}} = \frac{4bm}{d}\sqrt{1 - \frac{dd}{mm}}$, whence the ſeventeenth Equation becomes

	18	$ss - \frac{4sbm}{da} - 2sa + \frac{4bma}{da} = \frac{4bm}{d}\sqrt{1 - \frac{dd}{mm}}$, by which Means we have saved the Trouble of squaring the seventeenth Step.
$18 \times da$	19	$ssda - 4sbm - 2dsaa + 4bma = 4bma\sqrt{1 - \frac{dd}{mm}}$

The Equation being now cleared of its Fractions, it appears *quadratic*, for the Powers of a are only to the first and second Power, and the Surd is Part of one of the Co-efficients of a.

$19 \pm$	20	$2dsaa + 4bma\sqrt{1 - \frac{dd}{mm}} : -4bma - ssda = -4sbm$
$20 \div 2ds$	21	$aa + \frac{4bma\sqrt{1 - \frac{dd}{mm}} : -4bma - ssda}{2ds} = -\frac{4sbm}{2ds} = -\frac{2bm}{d}$
		Now $\frac{4bm\sqrt{1 - \frac{dd}{mm}} : -4bm - ssd}{2ds}$ being a known Quantity, and $= -45$, put $x = -45$.
then	22	$aa - xa = -\frac{2bm}{d}$
22 c □	23	$aa - xa + \frac{xx}{4} = \frac{xx}{4} - \frac{2bm}{d}$
23 ω 2	24	$a - \frac{x}{2} = \sqrt{\frac{xx}{4} - \frac{2bm}{d}}$
$24 + \frac{x}{2}$	25	$a = \frac{x}{2} \pm \sqrt{\frac{xx}{4} - \frac{2bm}{d}}$ the Equation being *ambiguous* by the 22d Step.

Whence we shall find $a = 27.21 = AC$.

And by the third Equation $\frac{2bm}{da} = 17.78 = BA$.

And by the thirteenth Equation $s - \frac{2bm}{da} - a = 40.01 = BC$.

 This

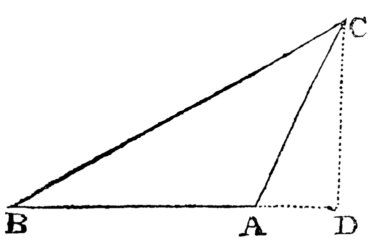
This being the most difficult Solution we have yet had, a Review or Summary Account of the Operation may not be useless to the Learner, in giving him some Idea how to begin and form a Judgment in such Cases.

Now because the Angles of the Triangle A C D are known, we have the *Ratio* of the Sides given, whence assuming C D as known, I find a proportional Number for A C, and from thence I can express C D and A C in Symbols, and C D being considered as the Perpendicular to the Triangle B A C, of which the Base is B A, from the Rule for finding the Area of a Triangle, I obtain an Expression for B A; and I express A D in Symbols, from knowing A C and C D, and adding A D to B A, I have Expressions for B D and D C, each of which being squared, their Sum is equal to the Square of B C.

Then from some other *Data* I find an Expression for B C, and because the Sum of the Sides is given, and having Expressions for the two Sides B A and A C, it is easy to find an Expression for B C, as at the thirteenth Step, which being squared, is made equal to the former Square of B C, and the Equation is reduced, as in the Work.

This and several other Questions are taken from Sir ISAAC NEWTON, the perpetual and everlasting Honour, Ornament, and Glory of our Nation; and I have only endeavoured to accommodate his Solutions to the Learner, in explaining them in a more copious Manner.

Question 98. *In the Triangle* A B C, *there is given the Altitude* C D $= 7$, *the Base* A B $= 10$, *and the Sum of the Sides* B C $+$ A C $= 23$. *To find the Sides of the Triangle?*



Because the three Sides of the Triangle B A C will be easily expressed in Symbols, and the Triangle B C D being right-angled, we shall easily find to what B D is equal.

Again, as the Triangle A C D is right-angled, and the Sides A C and C D are known in Symbols, therefore A D is known in Symbols.

Now if from B D before found in Symbols, we substract B A, there remains another Value for A D, which being made equal to the former, we have an Equation, which is sufficient, if we use but one unknown Quantity.

And

And as here will be a new Method of expressing the Quantities sought, I refer the Reader to Question 41, where at Step 8. he will find, that in any two Numbers, if their Difference is substracted from their Sum, and the Remainder divided by 2, the Quotient will be equal to the lesser; and at the same Question, if he exterminates e and finds a, or the greater Number, it will be equal to the Sum and Difference of the two Numbers added together and divided by 2.

Therefore put $x = CD = 7$, $b = BA = 10$, $c =$ half the Sum of the Sides $BC + AC = 11.5$ and $a =$ half their Difference, then the greater Side or $BC = c + a$, and the lesser Side or $AC = c - a$, now in the Triangle BCD

by 47 . e. 1	1	$\sqrt{cc+2ca+aa-xx} = BD$, for $BC = c+a$, and $CD = x$
		And in the Triangle ACD,
by 47 . e. 1	2	$\sqrt{cc-2ca+aa-xx} = AD$, for $AC = c-a$, and $CD = x$
but	3	$BA = b$
1 − 3	4	$\sqrt{cc+2ca+aa-xx} : -b = BD - BA = AD$
2 . 4	5	$\sqrt{cc-2ca+aa-xx} = \sqrt{cc+2ca+aa-xx} - b$
5 ⊙ 2	6	$cc-2ca+aa-xx = cc+2ca+aa-xx - 2b\sqrt{cc+2ca+aa-xx} : +bb$
		By transposing the Quantities which destroy one another
we have	7	$-4ca = -2b\sqrt{cc+2ca+aa-xx} : +bb$
7 ±	8	$2b\sqrt{cc+2ca+aa-xx} = bb+4ca$
8 ⊙ 2	9	$4bbcc + 8bbca + 4bbaa - 4bbxx = bbbb + 8bbca + 16ccaa$
9 − 8bbca	10	$4bbcc + 4bbaa - 4bbxx = bbbb + 16ccaa$
10 − bbbb	11	$16ccaa = 4bbcc + 4bbaa - 4bbxx - bbbb$
11 − 4bbaa	12	$16ccaa - 4bbaa = 4bbcc - 4bbxx - bbbb$
12 ÷ 16cc−4bb	13	$aa = \frac{4bbcc - bbbb - 4bbxx}{16cc-4bb} = \frac{bb}{4} - \frac{bbxx}{4cc-bb}$. See Page 276.
13 ɯ 2	14	$a = \sqrt{\frac{bb}{4} - \frac{bbxx}{4cc-bb}} = b\sqrt{\frac{1}{4} - \frac{xx}{4cc-bb}} = 3.69$

Whence $c+a=15.19=$ BC, and $c-a=7.81=$ AC.

If the Learner ſhould be perplexed to ſee the Contractions at the thirteenth and fourteenth Steps, they may be illuſtrated thus

$$\frac{4bbcc-bbbb-4bbxx}{16cc-4bb}=\frac{4bbcc-bbbb}{16cc-4bb}-\frac{4bbxx}{16cc-4bb}$$

for it is the ſame Thing whether the Quantities that compoſe the Numerator, are placed ſucceſſively one after another like one continued Fraction, or placed ſeparately and diſtinctly, like different Fractions, the Quantities that compoſe the Denominator being placed under each diſtinct Numerator.

$$\text{But } 16cc-4bb)\ 4bbcc-bbbb\left(\frac{bb}{4}\right.$$
$$\frac{4bbcc-bbbb}{0}$$

The Quotient Quantity is bb, and as the Co-efficients of the Diviſor are reſpectively four Times more than thoſe of the Dividend, therefore under the Quotient Quantity bb place 4, and $\frac{bb}{4}$ is the Quotient exact.

And this Fraction $\frac{4bbxx}{16cc-4bb}=\frac{bbxx}{4cc-bb,}$ for it is only dividing the Co-efficients by 4, therefore the Contractions are as at the thirteenth Step.

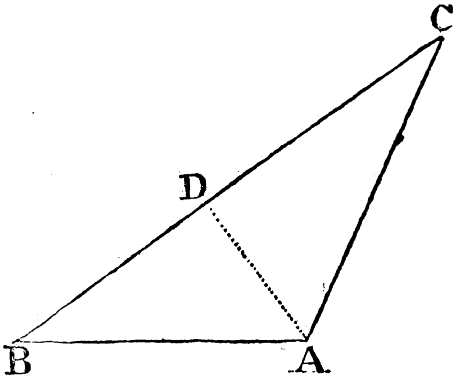
The Contractions at the fourteenth Step, ariſe from its being bb in all the Terms under the radical Sign, for it is only placing b the ſquare Root of bb without the radical Sign, by which means $\sqrt{\frac{bb}{4}}$, or $\sqrt{\frac{1\,bb}{4}}$ is $b\sqrt{\frac{1}{4}}$.

Queſtion 99. *In the Triangle* B C A, *there is given the Baſe* A B $=6$, *the Sum of the Sides* A C $+$ B C $=18$, *and the vertical Angle at* C $=30°:00'$. *To find the Sides* A C *and* B C.

Let fall the Perpendicular A D, and in the Triangle A C D, becauſe the Angle at C is given, all the Angles of that Triangle are known, and therefore the Ratio of the Sides is known, by which means we can get an Expreſſion for C D.

And

And becauſe A D is a Perpendicular that falls within the Triangle, and the Angle at C is acute, therefore by 13. *e.* 2, B C ſquared added to A C ſquared, is equal to B A ſquared added to the Product of 2 B C x C D, from whence we ſhall have another Expreſſion for C D. Now if we can expreſs the Sides of the Triangle with one unknown Quantity, this Equation between the two Values of C D will be ſufficient.



In the Triangle A C D, becauſe the Angle at C is known, and A D being a Perpendicular to C B, all the Angles of the Triangle A C D are known, therefore aſſuming C D = 1, by Trigonometry,

As the Sine of the Angle C A D	-	60° : 00′ -	9.937531
Is to the Log. of the Side C D	-	1. -	0.000000
So is the Sine of the Angle C D A	-	90 : 00 -	10.000000
			10.000000
			9.937531
To the Log. of the Side A C	-	1.15 - -	0.062469

Hence we know that as 1.15 is to 1, ſo is A C to C D.

Then let A B = 6 = x, half the Sum of the Sides A C + B C = 9 = b, and half their Difference = a, then as in the laſt Queſtion, the greater Side or B C = $b + a$, and the leſſer Side or A C = $b - a$, $d = 1.15$ $n = 1$.

Becauſe A C is to C D, as 1.15 is to 1.

Therefore	1	$d : n :: b - a : \frac{nb - na}{d} = CD$
by 13. *e*. 2	2	$bb + 2ba + aa + bb - 2ba + aa = xx + \overline{2b + 2a} \times CD$
$2 - xx$	3	$2bb + 2aa - xx = \overline{2b + 2a} \times CD$
$3 \div 2b + 2a$	4	$\frac{2bb + 2aa - xx}{\overline{2b + 2a}} = CD$

The ſhort Line over the two Quantities $2b + 2a$ in the ſecond and third Equations, ſignifies they are both to the multiplied into CD, otherwiſe there would be no Diſtinction whether CD is to be multiplied into $2a$ only, or all the Quantities on that Side of the Equation.

Now make an Equation between the two Values of CD found at the firſt and fourth Equations.

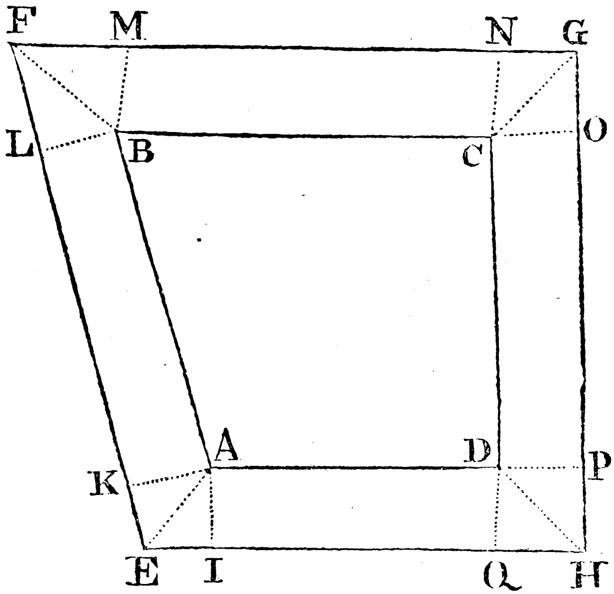
1 . 4	5	$\frac{2bb + 2aa - xx}{\overline{2b + 2a}} = \frac{nb - na}{d}$
$5 \times 2b + 2a$	6	$2bb + 2aa - xx = \frac{2nbb - 2bna + 2bna - 2naa}{d}$
$6 \times d$	7	$2dbb + 2daa - dxx = 2nbb - 2naa$
$7 + 2naa$	8	$2naa + 2dbb + 2daa - dxx = 2nbb$
$8 + dxx$	9	$2naa + 2dbb + 2daa = 2nbb + dxx$
$9 - 2dbb$	10	$2naa + 2daa = 2nbb + dxx - 2dbb$
$10 \div 2n + 2d$	11	$aa = \frac{2nbb + dxx - 2dbb}{2n + 2d}$
11 ω 2	12	$a = \sqrt{\frac{2nbb + dxx - 2dbb}{2n + 2d}} = 1.99$

Whence $BC = b + a = 10.99$ and $AC = b - a = 7.01$

Queſtion 100. *In the Fiſh Pond* ABCD, *there is given the Side* AD = 30, DC = 35, CB = 40, *and* AB = 38; *the Angle at* A = 113°, *the Angle at* B = 60°, *the Angle at* C = 100°, *and the Angle at* D = 87°, *and the Fiſh Pond is to be ſurrounded with an Area of* 700, *and every where of the ſame Breadth. To find the Breadth of the Walk?*

Suppoſing the Walk to be drawn round the Pond as in the Figure, let fall the Perpendiculars AK, BL, BM, CN, CO, DP, DQ and AI, by which the Walk is divided into four Parallellograms AKLB, BMNC, COPD, DQIA, and

and into four Trapezia A I E K, B L F M, C N G O and D P H Q, and the Area of these four Parallellograms and four Trapezia is equal to the given Area of the Walk.



Let the Breadth of the Walk be a, and the Sum of the Sides $AD + DC + CB + BA = 143 = b$, then the Area of the four Parallellograms will be $= b\,a$. Let $x = 700$.

Draw A E, B F, C G and D H, because the Triangles A I E and A K E are equal, therefore the Angle A E K and A E I are equal, and each of these Angles are equal to half the Angle at A which is 113°, hence the Angle A E I is $56^\circ : 30'$.

Then in the Triangle A E I all the Angles are known, and consequently from plain Trigonometry, we can find the Ratio of the Sides E I and I A, for assuming E I to be *Unity*, or 1, we have

As the Sine of the Angle E A I -	$33^\circ : 30'$ -	9.741889
Is to the Log. of the Side E I -	1.0 - -	0.000000
So is the Sine of the Angle A E I -	$56^\circ : 30'$ -	9.921107
		9.921107
		9.741889
To the Log. of the Side A I - -	1.51 - -	0.179218

Hence

Hence we know that A I is to E I as 1.51 is to 1.

Then let $d = 1.51$ and $e = 1$.

Hence $d : e :: a : \frac{ea}{d} = EI$, which being the Base of the Triangle E I A,

Hence	2	$\frac{ea}{d} \times \frac{a}{2} = \frac{eaa}{2d}$ = the Area of the (Triangle E I A, and
$2 \times \overline{2}$	3	$\frac{eaa}{d}$ = the Area of the Trapezium (E I A K.

Now in the Trapezium B L F M, becaufe the Angle B is 60°, we have the Angle L F B = 30°, for the fame Reafons as before; whence in the Triangle L F B, if we affume B L to be *Unity*, or 1, we fhall find the proportional Number for F L to be 1.73 hence as 1 is to 1.73 fo is B L to L F, let $f = 1.73$

Then	4	$e : f :: a : \frac{af}{e} = LF$
$4 \times \frac{a}{2}$	5	$\frac{af}{e} \times \frac{a}{2} = \frac{aaf}{2e}$ = the Area of the (Triangle B L F.
$5 \times \overline{2}$	6	$\frac{aaf}{e}$ = the Area of the Trapezium (B L F M.

Again, in the Trapezium C N G O, becaufe the Angle at C is 100°, the Angle C G N is 50°, for the fame Reafon as before; and affuming *Unity* for N G, we fhall find 1.19 to be the proportional Number for N C, whence we know that C N and N G are as 1.19 is to 1, let $g = 1.19$

Then	7	$g : e :: a : \frac{ea}{g} = NG$, which being the Bafe of the Triangle C N G,
Hence	8	$\frac{ea}{g} \times \frac{a}{2} = \frac{eaa}{2g}$ = the Area of the (Triangle C N G.
$8 \times \overline{2}$	9	$\frac{eaa}{g}$ = the Area of the Trapezium (C N G O.

Laftly, in the Trapezium D P H Q, becaufe the Angle D is 87°, the Angle D H P is 43° : 30′, for the fame Reafon as before;

before; and assuming *Unity* for D P we shall find 1.07 to be the proportional Number for P H, hence we know that as 1 is to 1.07 so is D P to P H, let $s = 1.07$

Then	10	$e : s :: a : \frac{as}{e} =$ P A, then as before
$10 \times \frac{a}{2}$	11	$\frac{as}{e} \times \frac{a}{2} = \frac{aas}{2e}$ the Area of the Triangle DPH, hence
11×2	12	$\frac{aas}{e} =$ the Area of the Trapezium D P H Q.
		But it was before found that
	13	$ba =$ the Area of the four Parallelograms.

Now collect the Area of the Trapezia and Parallelograms into one Sum, and make them equal to the given Area of the Walk.

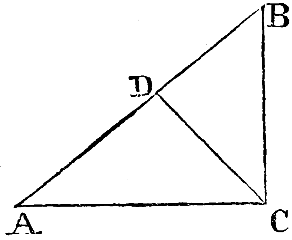
$3+6+9+12+13$	14	$\frac{eaa}{d} + \frac{aaf}{e} + \frac{eaa}{g} + \frac{aas}{e} + ba = x = 700$
substitute	15	$p = \frac{e}{d} + \frac{f}{e} + \frac{e}{g} + \frac{s}{e}$, the Coefficients of $aa = 4.302$
$14 . 15$	16	$paa + ba = x$
$16 \div p$	17	$aa + \frac{ba}{p} = \frac{x}{p}$
$17\ c\ \square$	18	$aa + \frac{ba}{p} + \frac{bb}{4pp} = \frac{bb}{4pp} + \frac{x}{p}$
$18\ \omega\ 2$	19	$a + \frac{b}{2p} = \sqrt{\frac{bb}{4pp} + \frac{x}{p}}$
$19 - \frac{b}{2p}$	20	$a = \sqrt{\frac{bb}{4pp} + \frac{x}{p}} : - \frac{b}{2p} = 4.35$ the Breadth of the Walk.

Because the Angles and Sides of the Fish Pond are given, the Figure may be drawn; but for the Ease of the Numerical Calculation I have chose such Numbers, as will not exactly agree with a Geometrical Figure.

Question 101. *In the right-angled Triangle* A B C, *given the Perimeter or Sum of the Sides* AC + C B + A B = 24, *and the*

 Perpendicular

Perpendicular CD $=4.75$ *let fall from the Right-angle at* C. *upon the Hypothenuse* A B. *To find the Sides of the Triangle?*



Let CD $= b = 4.75$ $AB + BC + CA = x = 24$, $AB = a$, then the Sum of two of the Sides, or $AC + CB = x - a$, and as at Queſtion 98 let $y =$ the Difference of the ſame two Sides A C and C B. And becauſe the greater Number or Leg is equal to the Sum and Difference of the two Numbers or Legs divided by 2, as in the laſt Queſtion, therefore A C the greater Leg $= \frac{x - a + y}{2}$, and B C the leſſer Leg $= \frac{x - a - y}{2}$.

Having Expreſſions for A B, B C, and A C, the three Sides of the Triangle A B C, in which there are two unknown Quantities a and y, we muſt raiſe two Equations from the Properties of the Figure, and becauſe the Triangles A B C, B C D are ſimilar, therefore by

4 . *e* . 6	1	AB : BC : : AC : CD
In Symbols	2	$a : \frac{x - a - y}{2} : : \frac{x - a + y}{2} : b$
whence	3	$ab = \frac{xx - xa + xy - xa + aa - ay - xy + ay - yy}{4}$
that is	4	$ab = \frac{xx - 2xa + aa - yy}{4}$
		And becauſe the Triangle A C B is right-angled,
by 47 . *e* . 1	5	$\frac{xx - 2xa - 2xy + 2ay + aa + yy}{4} + \frac{xx - 2xa + 2xy - 2ay + aa + yy}{4} = aa$
5 contracted	6	$\frac{2xx - 4xa + 2aa + 2yy}{4} = aa$

Hence the two Equations which contain the Queſtion are the fourth and ſixth, and as y is only to the Square in each of them, find in both the Value of yy. But the ſixth Equation becomes

7 | xx

	7	$\dfrac{xx-2xa+aa+yy}{2}=aa$
7×2	8	$xx-2xa+aa+yy=2aa$
$8 \pm$	9	$yy=aa+2xa-xx$
4×4	10	$4ab=xx-2xa+aa-yy$
$10+11$	11	$yy=xx-2xa+aa-4ab$
$9 . 11$	12	$aa+2xa-xx=xx-2xa+aa-4ab$
$12-aa$	13	$2xa-xx=xx-2xa-4ab$
$13+4ab$	14	$2xa+4ab-xx=xx-2xa$
$14+2xa$	15	$4xa+4ab-xx=xx$
$15+xx$	16	$4xa+4ab=2xx$
$16\div 4x+4b$	17	$a=\dfrac{2xx}{4x+4b}=\dfrac{xx}{2x+2b}=10=AB.$
		Now a being found, therefore
$9 \text{ ш } 2$	18	$y=\sqrt{aa+2xa-xx}=2.$
thence	19	$AC=\dfrac{x-a+y}{2}=8.$
and	20	$BC=\dfrac{x-a-y}{2}=6.$

Queſtion 102. *In the right-angled Triangle* A B C, *given the Hypothenuſe* A B $=10$, *and the Sum of the Sides and Perpendicular* C D, *that is* A C + C B + C D $=18.75$ *To find the Sides* A C *and* B C? *Vide laſt Figure.*

Let A B $=b=10$, $x=18.75$ C D $=a$, then A C + C B $=x-a$; now put $y=$ the Difference between the Legs A C and C B, then as in the two laſt Queſtions A C, or the greater Leg is $\dfrac{x-a+y}{2}$, and the leſſer Leg, or C B $=\dfrac{x-a-y}{2}$.

Having expreſſed the Sides of the Triangle A B C in Symbols, in which there are two unknown Quantities a and y, we muſt raiſe two Equations from the Properties of the Figure, and becauſe A B C is a right-angled Triangle, therefore

by 47 . e . 1	1	$\dfrac{xx-2xa+2xy-2ay+aa+yy}{4}$
		$+\dfrac{xx-2xa-2xy+2ay+aa+yy}{4}=bb$

 The

The two Triangles ABC and CBD being ſimilar, therefore

by 4 . *e* . 6	2	AB : AC : : CB : CD, that is in Symbols,
	3	$b : \frac{x-a+y}{2} :: \frac{x-a-y}{2} : a$
3 ∵	4	$ba = \frac{xx-xa+xy-xa+aa-ay-xy+ay-yy}{4}$
		Hence the Queſtion is contained in the firſt and fourth Equations.
1 contracted	5	$\frac{xx-2xa+aa+yy}{2} = bb$
4 contracted	6	$ba = \frac{xx-2xa+aa-yy}{4}$

Now in both theſe Equations find the Value of yy, there being no other Power of y.

$5 \times \overline{2}$	7	$xx-2xa+aa+yy=2bb$
$7 \pm$	8	$yy=2bb-aa+2xa-xx$
$6 \times \overline{4}$	9	$4ba=xx-2xa+aa-yy$
$9 \pm$	10	$yy=xx-2xa+aa-4ba$
8 . 10	11	$xx-2xa+aa-4ba=2bb-aa+2xa-xx$
$11+aa$	12	$xx-2xa+2aa-4ba=2bb+2xa-xx$
$12-xx$	13	$-2xa+2aa-4ba=2bb+2xa-2xx$
$13-2xa$	14	$2aa-4xa-4ba=2bb-2xx$
$14 \div \overline{2}$	15	$aa-2xa-2ba=bb-xx$
		Becauſe xx is greater than bb, therefore the Equation is *ambiguous.*
ſubſtitute	16	$-z=-2x-2b=-57.5$
then	17	$aa-za=bb-xx$
17 *c* □	18	$aa-za+\frac{zz}{4}=bb-xx+\frac{zz}{4}$
18 *w* 2	19	$a-\frac{z}{2}=\sqrt{bb-xx+\frac{zz}{4}}$
$19+\frac{z}{2}$	20	$a=\frac{z}{2}\pm\sqrt{bb-xx+\frac{zz}{4}}=4.78$ (or 52.72
by the 10th Step	21	$y=\sqrt{xx-2xa+aa-4ba}=1.99$

Then

Then $AC = \frac{x - a + y}{2} = 7.98$ and $CB = \frac{x - a - y}{2} = 6$.

In the above Equation where $a = 4.78$ or 52.72 the Value of a muſt be 4.78 for it cannot be 52.72 as the Sum of the three Quantities is only 18.75.

Queſtion 103. *In the right-angled Triangle* ABC, *there is given the Sum of the Sides* $AC + BC = 14$, *and the Perpendicular* $CD = 4.75$ *To find the Sides of the Triangle?* See Figure, Queſtion 101.

Let $AC + BC = x = 14$, $AC - BC = y$ the Difference of the Sides, then, as in the preceding Queſtions, the greater Side or $AC = \frac{x + y}{2}$, and the leſſer Side or $BC = \frac{x - y}{2}$.

Put $AB = a$ and $DC = b = 4.75$

Having expreſſed all the Sides of the Triangle ABC in Symbols, amongſt which two are unknown, *viz.* a and y, we muſt raiſe two Equations from the Figure, then becauſe the Triangle ABC is right-angled, therefore

by 47 . *e* . 1	1	$\frac{xx + 2xy + yy}{4} + \frac{xx - 2xy + yy}{4} = aa$
that is	2	$\frac{xx + yy}{2} = aa$
		Becauſe the Triangles ABC and CBD are ſimilar,
by 4 . *e* . 6	3	$AB : AC :: CB : CD$
in Symbols	4	$a : \frac{x + y}{2} :: \frac{x - y}{2} : b$
4 ∵	5	$ba = \frac{xx - yy}{4}$

Hence the Queſtion is contained in the ſecond and fifth Equations, and becauſe there are no other Powers of y but yy in either of thoſe two Equations, find the Value of yy in both Equations.

$2 \times \overline{2}$	6	$xx + yy = 2aa$
$6 - \overline{xx}$	7	$yy = 2aa - xx$
5×4	8	$4ba = xx - yy$
$8 + yy$	9	$yy + 4ba = xx$

$9 - 4ba$	10	$yy = xx - 4ba$
$7 \cdot 10$	11	$2aa - xx = xx - 4ba$
$11 \pm$	12	$2aa + 4ba = 2xx$
$12 \div 2$	13	$aa + 2ba = xx$
$13\ c\ \square$	14	$aa + 2ba + bb = xx + bb$
$14\ w\ 2$	15	$a + b = \sqrt{xx + bb}$
$15 - b$	16	$a = \sqrt{xx + bb} : - b = 10.03$ or neglecting the Fraction $AB = 10$.
$7\ w\ 2$	17	$y = \sqrt{2aa - 2x} = 2$

Then $AC = \dfrac{x+y}{2} = 8$, and $BC = \dfrac{x-y}{2} = 6$.

The same Question done in another Manner.

Let $AC + BC = x = 14$, $AC = a$, then $BC = x - a$, $CD = b = 4.75$ and because the Triangle ABC is right-angled, therefore $AB = \sqrt{xx - 2xa + 2aa}$.

Here we have Expressions for all the Sides of the Triangle, with only one unknown Quantity, and therefore one Equation will be sufficient. And as the Triangles ABC and CBD are similar, therefore

by $4 . e . 6$	1	$AB : AC :: CB : CD$
in Symbols	2	$\sqrt{xx - 2xa + 2aa} : a :: x - a : b$
$2 \because$	3	$b\sqrt{xx - 2xa + 2aa} = xa - aa$

Square both Sides of the Equation, the unknown Quantity being under the radical Sign.

$3 \odot 2$	4	$bbxx - 2bbxa + 2bbaa = xxaa - 2xaaa + aaaa$
		Ranging the Equation according to the Powers of the unknown Quantity.
$4 \pm$	5	$aaaa - 2xaaa + xxaa - 2bbaa + 2bbxa = bbxx$

Tho' the Equation here appears as if *adfected*, yet it may be resolved by *compleating the square*, as in Quadratics.

And to give the Learner a clear Idea how this is done, if he squares any three Quantities $m - n - z$, in the Square he will find six Terms, $mm + nn + zz - 2mn - 2mz + 2nz$, three

three being pure Powers of the Quantities squared, and the other three will be double Rectangles, or Products of these Quantities, and therefore any Expression that comes under these Circumstances, may have its square Root extracted.

Now $aaaa$ is the Square of - - - aa
And $xxaa$ is the Square of - - - xa

And $2xaaa$ is the double Rectangle, or Product of these Roots.
And $2bbaa$ is the double Rectangle, or Product of $bb \times aa$.
And $2bbxa$ is the double Rectangle, or Product of $bb \times xa$.

From hence it appears that the above Equation of five Quantities has two of them, $aaaa$ and $xxaa$, whose square Roots may be taken, and that the other three Quantities are double Rectangles of those two Roots, aa and xa, and a third Quantity bb, therefore multiply this Quantity bb by itself, and add the Product $bbbb$ to both Sides of the Equation, which makes it a compleat Square, thus,

	6	$aaaa - 2xaaa + xxaa - 2bbaa + 2bbxa + bbbb = bbbb + bbxx$
6 ω 2	7	$aa - xa - bb = \sqrt{bbbb + bbxx} = b\sqrt{bb + xx}$
$7 + bb$	8	$aa - xa = bb + b\sqrt{bb + xx}$
8 c □	9	$aa - xa + \frac{xx}{4} = \frac{xx}{4} + bb + b\sqrt{bb + xx}$
9 ω 2	10	$a - \frac{x}{2} = \sqrt{\frac{xx}{4} + bb + b\sqrt{bb + xx}}$
$10 + \frac{x}{2}$	11	$a = \frac{x}{2} + \sqrt{\frac{xx}{4} + bb + b\sqrt{bb + xx}} = 18.9 =$ A C, a different Value of what it had before, for then it was only 8.

To explain this to the Learner, if he extracts the Square Root of $aa - 2xa + xx$, he will find it to be $a - x$, or $x - a$, the double Rectangle, *viz.* $2ax$ having the Sign — we are sure either a, or x must be *negative*; but in this Case we are to determine which is to be *negative* by the Consequences that follow, for if there follows an Impossibility in supposing $a - x$ to be the Root, then the Root must be $x - a$.

To apply this to the Square before us at the sixth Step, *viz.* $aaaa - 2xaaa + xxaa - 2bbaa + 2bbxa + bbbb$.

3 Now

Now the ſquare Root of $aaaa$ is - - aa
And the ſquare Root of $xxaa$ is - - xa
And the ſquare Root of $bbbb$ is - - bb

But as $2xaaa$ the double Rectangle, or Product of $aa \times xa$ has the Sign —, therefore it muſt be in the ſquare Root either $aa - xa$, or $xa - aa$; but as an Impoſſibility attends putting it $aa - xa$, we now put it $xa - aa$, and to determine what Sign bb muſt have in the Root, now the double Product $2bbaa$ having the Sign —, therefore it muſt be $+bb$ or bb, as $2bb \times -aa$ produces $-2bbaa$, then taking the

ſixth Equation	7	$aaaa - 2xaaa + xxaa - 2bbaa + 2bbxa + bbbb = bbbb + bbxx$
7 ɯ 2	8	$xa - aa + bb = \sqrt{bbbb + bbxx} = b\sqrt{bb + xx}$
		Becauſe aa is *negative* tranſpoſe it,
$8 + aa$	9	$aa + b\sqrt{bb + xx} = xa + bb$
$9 - xa$	10	$aa - xa + b\sqrt{bb + xx} = bb$
$10 - b\sqrt{bb + xx}$	11	$aa - xa = bb - b\sqrt{bb + xx}$

Here the Equation appears *quadratic*, and becauſe $-b\sqrt{bb + xx}$ is greater than bb, it is likewiſe *ambiguous*.

11 c □	12	$aa - xa + \frac{xx}{4} = \frac{xx}{4} + bb - b\sqrt{bb + xx}$
12 ɯ 2	13	$a - \frac{x}{2} = \sqrt{\frac{xx}{4} + bb - b\sqrt{bb + xx}}$
$13 + \frac{x}{2}$	14	$a = \frac{x}{2} \pm \sqrt{\frac{xx}{4} + bb - b\sqrt{bb + xx}} = 7 \pm 1.11 = 8.11$ or $5.89 = AC.$

Queſtion 104. *In the right-angled Triangle* ABC, *there is given the Sum of the Legs* $AC + BC = 14$, *and the Sum of the Hypothenuſe and Perpendicular* $AB + CD = 14.75$ *To find the Sides of the Triangle?* See Figure, Queſtion 101.

Let $AC + BC = 14 = x$, $AB + CD = 14.75 = b$, $AC = a$, $AB = y$, then $BC = x - a$, and $CD = b - y$.

Having now expreſſed the Sides of the Triangle in Symbols, in which there are two that are unknown, therefore raiſe two Equations from the Properties of the Figure.

And

And because the Triangle ABC is right-angled, therefore by

47 *e*. 1	1	$xx - 2xa + 2aa = yy$

And because the Triangles ABC and CBD are similar, therefore by

4 *e*. 6	2	AB : AC :: BC : CD
in Symbols	3	$y : a :: x - a : b - y$
3 ∵	4	$by - yy = xa - aa$

Now both the unknown Quantities being to the first and second Power, in the fourth Equation, and it being yy only in the first Equation, and these two Equations containing the Conditions of the Question, find the Value of yy in each Equation.

$4 \pm$	5	$yy = by + aa - xa$
1 . 5	6	$by + aa - xa = xx - 2xa + 2aa$
$6 - aa$	7	$by - xa = xx - 2xa + aa$
$7 + xa$	8	$by = xx - xa + aa$
$8 \div b$	9	$y = \frac{xx - xa + aa}{b}$

Raise this Equation to the second Power, and make it = to the first Equation, as there it is only yy; whereas in the fourth Equation, if we were to exterminate y, we must use the Values of y and yy.

9 Ⓖ 2	10	$yy = \frac{xxxx - 2axxx + 2aaxx - 2aaax + aaxx + aaaa}{bb}$
1 . 10	11	$\frac{xxxx - 2axxx + 2aaxx - 2aaax + aaxx + aaaa}{bb} = xx - 2xa + 2aa$
$11 \times bb$	12	$xxxx - 2axxx + 2aaxx - 2aaax + aaxx + aaaa = xxbb - 2bbxa + 2bbaa$

Transposing and ranging the Equation, according to the highest Dimensions of the unknown Quantity.

$12 \pm$	13	$aaaa - 2xaaa + 2aaxx - 2bbaa + aaxx - 2axxx + 2bbxa + xxxx - xxbb = 0$

Tho' the Equation now appears to be *adfected*, yet the square Root may be compleated, as in the last.

To show the Learner how this is to be done, if he squares any four Quantities, (for the Root of the above Equation will consist

of fo many Quantities) he will find ten Terms in the Square, four of which are pure Powers of the Quantities that were fquared, and the other fix will be double Rectangles of thofe Quantities, of which each particular Root will conftitute a Part of three of the Rectangles.

Now in the above Equation $aaaa$, $aaxx$, $xxxx$,
Are the Squares of - - aa, ax, xx,
And the Quantities - - $2xaaa$, $2aaxx$, $2axxx$,
are the double Rectangles of thofe Parts, or Roots.

And by examining $2bbaa$, $2bbxa$, $xxbb$, the remaining Terms in the above Equation, the firft two are double Rectangles of $bb \times aa$ and $bb \times ax$, but the laft Term is only a fingle Rectangle of $bb \times xx$, therefore to compleat the Square there wants $-xxbb$, which when added to $-xxbb$, will make that a double Rectangle of $bb \times xx$, and as we have no pure Power of bb, which being fquared is $bbbb$, hence if we add $-bbxx+bbbb$ to our Equation, we fhall make it a Square, therefore

$$14 \quad \left| \begin{array}{l} aaaa - 2xaaa + aaxx - 2bbaa + aaxx \\ -2axxx + 2bbxa + xxxx - 2xxbb \\ +bbbb = bbbb - xxbb \end{array} \right.$$

Before we proceed, perhaps the Learner might have obferved that $xxbb$ is the Square of xb, and therefore might fuppofe that to be one of the Roots, but then he will find xb to make a Part only of two of the Rectangles, whereas, if it had been one of the Roots, it would have made a Part of three of the Rectangles. Befides, if xb had been one of the Roots, it muft have had the Sign $+$ in the Square, by Art. 34.

$$14 \; \omega \; 2 \quad \Big| \quad 15 \quad \left| \begin{array}{l} aa - ax + xx - bb = \sqrt{bbbb - xxbb} \\ = b\sqrt{bb - xx} \end{array} \right.$$

The Manner of extracting the Root is thus, I firft extract the fquare Root of $aaaa$, which is aa, then the fquare Root of $aaxx$; the next pure Power is ax, and to determine whether ax muft have the Sign $+$ or $-$, obferve the Sign of the double Rectangle of thefe two Roots, *viz.* of $2xaaa$, which becaufe it is $-$, I therefore in the Root make it $-ax$.

The next pure Power is $xxxx$, whofe Root is xx, then obferve the Sign of the double Rectangle of this and one of the two former Roots, as of $2aaxx$, which being $+$, and the Root aa being $+$, therefore in the Root make it $+xx$.

The

The last pure Power is $bbbb$, whose Root is bb, then observe the Sign of the double Rectangle of this and one of the former Roots, as the last Root xx, but the double Rectangle of these is $2xxbb$, which being *negative*, and the Sign of xx being $+$, therefore place the Sign $-$ before bb.

$15 + bb$	16	$aa - ax + xx = bb + b\sqrt{bb - xx}$
$16 - xx$	17	$aa - ax = bb - xx + b\sqrt{bb - xx}$
$17\ c\ \square$	18	$aa - ax + \frac{xx}{4} = bb - \frac{3xx}{4} + b\sqrt{bb - xx}$, for $\frac{xx}{4} - xx = -\frac{3xx}{4}$
$18\ \omega\ 2$	19	$a - \frac{x}{2} = \sqrt{bb - \frac{3xx}{4} + b\sqrt{bb - xx}}$
$19 + \frac{x}{2}$	20	$a = \frac{x}{2} + \sqrt{bb - \frac{3xx}{4} + b\sqrt{bb - xx}}$

$= 18.79 = AC$, which is impossible, for $AC + BC = 14$ by the Question, consequently AC cannot be 18.79

This impossible Conclusion is owing to taking the Root of the Equation at the fifteenth Step, for as $-aa \times -aa$ produces $aaaa$, as well as $aa \times aa$, therefore in the Extraction of such Roots, it is doubtful whether the Root is $-aa$, or aa, let us now make a new Extraction, and suppose it to be $-aa$.

$14\ \omega\ 2$	21	$-aa + ax - xx + bb = \sqrt{bbbb - xxbb} = b\sqrt{bb - xx}$

Having supposed the Root of $aaaa$ the first pure Power to be $-aa$, I go to the next pure Power, which is $aaxx$, whose Root is ax; but to determine its Sign, observe the Sign of the double Rectangle of these two Roots, *viz.* of $2aaax$, which being $-$, I therefore make it $+ax$, as $-$ into $+$ produces $-$.

The next pure Power is $xxxx$, whose Root is xx, then observe the Sign of the double Rectangle of this, and either of the two former Roots, as of ax, now the Sign of $2axxx$ is $-$, therefore in the Root make it $-xx$, for $+ax \times -xx$ produces $-axxx$.

The last pure Power is $bbbb$, whose Root is bb, and observe the Sign of the double Rectangle of this, and either of the other Roots, as suppose the last, the double Rectangle of these two

Roots is $2bbxx$, which being $-$, therefore make it $+bb$, as $-xx \times bb$ gives $-xxbb$.

Now tranſpoſe aa, it being *negative.*

$21 + aa$	22	$aa + b\sqrt{bb - xx} = ax - xx + bb$
$22 - b\sqrt{bb - xx}$	23	$aa = bb + ax - xx - b\sqrt{bb - xx}$
$23 - ax$	24	$aa - ax = bb - xx - b\sqrt{bb - xx}$

Here the Equation is *quadratic*, and becauſe $-xx - b\sqrt{bb - xx}$ is greater than bb, it is therefore *ambiguous.*

$24\ c\ \square$	25	$aa - ax + \frac{xx}{4} = bb - \frac{3xx}{4} - b\sqrt{bb - xx}$
$25\ \text{ɯ}\ 2$	26	$a - \frac{x}{2} = \sqrt{bb - \frac{3xx}{4} - b\sqrt{bb - xx}}$
$26 + \frac{x}{2}$	27	$a = \frac{x}{2} \pm \sqrt{bb - \frac{3xx}{4} - b\sqrt{bb - xx}}$

$$\begin{array}{r} x = 14 \\ x = 14 \\ \hline 56 \\ 14 \\ \hline xx = 196 \end{array} \qquad \begin{array}{r} b = 14.75 \\ b = 14.75 \\ \hline 7375 \\ 10325 \\ 5900 \\ 1475 \\ \hline 217.5625 = bb \\ -\ 196 \qquad = -xx \\ \hline 21.5625 = (4.64 = \sqrt{bb - xx} \\ 16 \\ \hline 86)\ 556 \\ 516 \\ \hline 924)\ 4025 \\ 3696 \\ \hline 329 \end{array}$$

14.75

$$14.75 = b$$
$$4.64 = \sqrt{bb - xx}$$

5900
8850
5900

$$68.4400 = b\sqrt{bb - xx}$$
$$147. = \frac{3xx}{4}$$

$$215.44 = b\sqrt{bb - xx} + \frac{3xx}{4}$$

$$xx = 196$$
$$3$$
$$4)\ 588 = 3xx$$
$$147 = \frac{3xx}{4}$$

$$217.5625 = bb$$
$$-215.44 = -\frac{3xx}{4} - b\sqrt{bb - xx}$$

$$2.1225\ (1.46 \text{ nearest} = \sqrt{bb - \frac{3xx}{4} - b\sqrt{bb - xx}}$$

1
24) 112
96
286) 1625

$$7 = \frac{x}{2}$$

$$\pm 1.46 = \sqrt{bb - \frac{3xx}{4} - b\sqrt{bb - xx}}$$
$$8.46 = a = \text{AC.}$$
$$\text{or } 5.54 = a = \text{AC.}$$

But if $a = 8.46$ then by the ninth Step $y = \frac{xx - xa + aa}{b}$

$$a = 8.46$$
$$x = 14.$$

3384
846

$$ax = 118.44$$

8.46

$$\begin{array}{r l}
8.46 & = a \\
8.46 & = a \\
\hline
5076 & \\
3384 & \\
6768 & \\
\hline
71.5716 & = aa \\
196 & = xx \\
\hline
267.5716 & = aa + xx \\
-118.44 & = -ax \\
\hline
\end{array}$$

$$b = 14.75)\ 149.1316\ (10.11 = y = \text{A B}.$$

$$\begin{array}{r}
1475 \\
\hline
1631 \\
1475 \\
\hline
1566 \\
1475 \\
\hline
91
\end{array}$$

Becauſe $a = \text{A C} = 8.46$ therefore $\text{B C} = x - a = 5.54$.

That theſe are the three Sides of a right-angled Triangle may be tried, by ſquaring and adding them, to ſee if they agree with the Property of the Figure.

$$\begin{array}{r}
5.54 \\
5.54 \\
\hline
2216 \\
2770 \\
2770 \\
\hline
30.6916 \\
71.5716 \\
\hline
102.2632 \\
102.2121 \\
\hline
.0511
\end{array}
\qquad
\begin{array}{r}
10.11 \\
10.11 \\
\hline
1011 \\
1011 \\
10110 \\
\hline
102.2121
\end{array}
\qquad
\begin{array}{r}
8.46 \\
8.46 \\
\hline
5076 \\
3384 \\
6768 \\
\hline
71.5716
\end{array}$$

.0511 the Difference which ariſes from the Inaccuracy of the Fractions.

But if the laſt Proceſs is too perplexing, the ſame Queſtion may be done otherwiſe, thus,

Let

Let $AC + BC = 14 = x$, and $a =$ the Difference between AC and BC, whence, as in the former Queſtions, the greater Leg or $AC = \frac{x+a}{2}$, and the leſſer Leg $BC = \frac{x-a}{2}$.

Again, put $AB + CD = 14.75 = b$, and the Difference between AB and $CD = y$, then for the Reaſons already mentioned $AB = \frac{b+y}{2}$, and $CD = \frac{b-y}{2}$.

Now becauſe the Triangle ACB is right-angled,

by 47 *e*. 1	1	$\frac{xx+2xa+aa}{4} + \frac{xx-2xa+aa}{4} = \frac{bb+2by+yy}{4}$

Becauſe the Triangles ACB and BCD are ſimilar, therefore

by 4 *e*. 6	2	$AB : AC :: BC : CD$
in Symbols	3	$\frac{b+y}{2} : \frac{x+a}{2} :: \frac{x-a}{2} : \frac{b-y}{2}$
$3 \because$	4	$\frac{bb-yy}{4} = \frac{xx-aa}{4}$ or $bb - yy = xx - aa$
from the firſt	5	$\frac{xx+aa}{2} = \frac{bb+2by+yy}{4}$

The Queſtion being contained in the fourth and fifth Equations, and there being no other Powers of a but aa in both thoſe Equations, exterminate that unknown Quantity.

$4 \pm$	6	$aa = xx + yy - bb$
5×4	7	$2xx + 2aa = bb + 2by + yy$
$7 - 2xx$	8	$2aa = bb + 2by + yy - 2xx$
$8 \div 2$	9	$aa = \frac{bb+2by+yy-2xx}{2}$
6 . 9	10	$xx + yy - bb = \frac{bb+2by+yy-2xx}{2}$
10×2	11	$2xx + 2yy - 2bb = bb + 2by + yy - 2xx$
$11 - yy$	12	$2xx + yy - 2bb = bb + 2by - 2xx$
$12 + 2bb$	13	$2xx + yy = 3bb + 2by - 2xx$
$13 - 2xx$	14	$yy = 3bb + 2by - 4xx$
$14 - 2by$	15	$yy - 2by = 3bb - 4xx$

Here

Here the Equation is *quadratic*, and ſince — $4xx$ is greater than $3bb$ it is *ambiguous.*

$15c\ \square$	16	$yy-2by+bb=bb+3bb-4xx$ $=4bb-4xx$
16 ա 2	17	$y-b=\sqrt{4bb-4xx}=2\sqrt{bb-xx}$
$17+b$	18	$y=b\pm2\sqrt{bb-xx}=14.75\pm9.3$ $=24.05$ or 5.45

But y cannot be 24.05 for the Sum of the Legs is only 14.75 therefore $y=5.45$

then by Step 6th | 19 | $a=\sqrt{xx+yy-bb}=2.85$

Then $AB=\frac{b+y}{2}=10.1$ $AC=\frac{x+a}{2}=8.42$ $BC=\frac{x-a}{2}=5.57$ which three Numbers nearly agree with the Property of the right-angled Triangle, but not exactly, becauſe of the Imperfection of the Fractions.

The Reader may obſerve, that in ſeveral of the Geometrical Queſtions, after Letters are put for one or more of the unknown Quantities, we then get Expreſſions for the other Parts of the Figure from its Properties, and therefore avoid uſing a greater Number of unknown Quantities, and in general the Solution of Queſtions are more neat and elegant, the fewer unknown Quantities are uſed in the Work.

The Method of reſolving Queſtions, which contain four Equations, and four unknown Quantities.

72. WHEN the Queſtion contains four Equations, and there are four unknown Quantities in each Equation; find the Value of one of the unknown Quantities in one of the given Equations, and for that unknown Quantity in the other three Equations write this Value of it, which then reduces the Queſtion to three Equations, and three unknown Quantities.

Then

Then find the Value of one of theſe three unknown Quantities in one of theſe three Equations, and for that unknown Quantity in the other two Equations write this Value of it, which reduces the Queſtion to two Equations, and two unknown Quantities.

Then find the Value of one of the unknown Quantities in each of theſe two Equations, and making theſe Equations equal to one another, we ſhall have an Equation with only one unknown Quantity, which being reduced, will anſwer the Queſtion.

Queſtion 105. *A Father gave* 1000l. *to his four Sons* A, B, C, D.

If A's *Share was added to twice* B's *Share, from which Sum ſubſtracting twice* C's *and* D's *Shares, there remains* 650 *Pounds:*

And if from A's *Share there is ſubſtracted three times* B's *Share, to the Remainder adding twice* C's *Share, from which Sum ſubſtracting five times* D's *Share, there remains* 400 *Pounds:*

But if to A's *Share there is added four times* B's *Share, from which Sum ſubſtracting three times* C's *Share, and to the Remainder adding ſix times* D's *Share, the Sum is* 1150 *Pounds. How much had each Son?*

Let $a =$ A's Share, $e =$ B's Share, $y =$ C's Share, $u =$ D's Share, $s = 1000$, $m = 650$, $n = 400$, $b = 1150$.

	1	$a + e + y + u = s$	By the Queſtion.
	2	$a + 2e - 2y - 2u = m$	
	3	$a - 3e + 2y - 5u = n$	
	4	$a + 4e - 3y + 6u = b$	
from the firſt	5	$a = s - u - y - e$	
5 . 2	6	$s + e - 3y - 3u = m$	
5 . 3	7	$s - 4e + y - 6u = n$	
5 . 4	8	$s + 3e - 4y + 5u = b$	

Here the Queſtion is reduced to three Equations, and three unknown Quantities.

from the ſixth	9	$e = m + 3u + 3y - s$
9 . 7	10	$s - 4m - 12u - 12y + 4s + y - 6u = n$
9 . 8	11	$s + 3m + 9u + 9y - 3s - 4y + 5u = b$

Here the Queſtion is reduced to two Equations, and two unknown Quantities.

10 contracted	12	$5s - 4m - 18u - 11y = n$
11 contracted	13	$-2s + 3m + 14u + 5y = b$
from the twelfth	14	$y = \frac{5s - 4m - 18u - n}{11}$
from the thirteenth	15	$y = \frac{b + 2s - 3m - 14u}{5}$
14 . 15	16	$\frac{b+2s-3m-14u}{5} = \frac{5s-4m-18u-n}{11}$
16 ×	17	$11b+22s+33m-154u=25s-20m-90u-5n$
17 ±	18	$64u = 11b - 3s - 13m + 5n$
18 ÷ 64	19	$u = \frac{11b - 3s - 13m + 5n}{64} = 50$, the Share of D.
then by Step 15th	20	$y = \frac{b + 2s - 3m - 14u}{5} = 100$, the Share of C.
and by Step 9th	21	$e=m+3u+3y-s=100$, the Share of B.
and by Step 5th	22	$a=s-u-y-e=750$, the Share of A.

And in the same Manner may any other Question in the like Circumstances be answered.

I shall now add a few Questions of a different Nature, and such as are generally first proposed to Learners, but as they require a little more Sagacity to express their Conditions, have hitherto been avoided, imagining the Learner is more perplexed to express, or find out the Equations resulting from such Questions, than to resolve the Equations; and therefore they were thought not so proper at the Beginning of this Work.

Question 106. *A Person bought two Horses* A *and* B, *which with the Trappings cost* 100 *Pounds:*

Now if the Trappings were laid on the Horse A, *both Horses were of equal Value:*

But if the Trappings be laid on the Horse B, *he will be double the Value of the Horse* A. *How much did each Horse cost?*

Let $b = 100$, $a =$ the Value of the Horse B and Trappings, then $b - a =$ the Value of the Horse A.

Now because the Horse B and Trappings are double the Value of the Horse A,

hence

hence	1	$a = 2b - 2a$ by the Question.
$1 + 2a$	2	$3a = 2b$
$2 \div 3$	3	$a = \frac{2b}{3} = \frac{200}{3} = 66\frac{2}{3}$ Pounds, the

Price of the Horse B and Trappings. Consequently $100 - 66\frac{2}{3} = 33\frac{1}{3}$ Pounds, the Price of the Horse A.

But to find what the Trappings cost, and by that Means to find the Price of the Horse B, let $y =$ the Price of the Trappings.

Now the Trappings taken from the Horse B, and laid upon the Horse A, both Horses being then of equal Value,

therefore	1	$33\frac{1}{3} + y = 66\frac{2}{3} - y$
$1 + y$	2	$33\frac{1}{3} + 2y = 66\frac{2}{3}$
$2 - 33\frac{1}{3}$	3	$2y = 33\frac{1}{3}$
$3 \div 2$	4	$y = 16\frac{2}{3}$ Pounds, the Price of the (Trappings.

Consequently $33\frac{1}{3} + 16\frac{2}{3} = 50$ Pounds, the Price of the Horse B.

Question 107. *A Labourer in 40 Weeks Labour saved 28 Crowns — the Pay of three Weeks, and found he had spent 36 Crowns + the Pay of eleven Weeks. How much did he receive a Week?*

Let $a =$ his weekly Pay.

Then he had saved Crowns - - - -	$28 - 3a$
And spent Crowns - - -	$36 + 11a$
	$64 + 8a$

And as the Sum of these two must be equal to what he received for his forty Weeks Labour,

therefore	1	$40a = 64 + 8a$
$1 - 8a$	2	$32a = 64$
$2 \div 32$	3	$a = 2$ Crowns, his weekly Pay.

Queſtion 108. *A Servant was hired for* 12 *Months, for which he was to have* 24 *Pounds with a Cloak; when he had ſerved* 8 *Months he has Leave to go away, and inſtead of his Wages receives a Cloak and* 13 *Pounds. How much did the Cloak coſt?*

Let $a =$ the Price of the Cloak, $b = 12$, $d = 24$, $m = 8$, $x = 13$.

Now $d + a$ is what he did receive for ſerving eight Months.

And as the Pay for eight Months was proportional to what he was to receive for twelve Months, therefore,

	1	$d + a : b :: x + a : m$ When any four Quantities, or Numbers, are in Geometrical Proportion, the Product of the Extreams and Means are equal,
therefore	2	$md + ma = bx + ba$
$2 \pm$	3	$ba - ma = md - bx$
$3 \div b - m$	4	$a = \dfrac{md - bx}{b - m} = 9$ Pounds, the Price of the Cloak.

Queſtion 109. *There is a Footman* A, *who goes* 6 *Miles a Day, and* 8 *Days after* B *follows him and goes* 10 *Miles a Day. In how many Days will* B *overtake* A?

Let $b = 6$, $d = 8$, $m = 10$, $a =$ the Number of Days B travels to overtake A, then as A begun to walk eight Days before B,

Hence the Number of Days that A travels, is -	$d + a$
And the Number of Miles A travels, is -	$bd + ba$
And the Number of Miles B travels, is - -	ma

But when B overtakes A, they muſt have travelled an equal Number of Miles.

Therefore	1	$ma = bd + ba$
$1 - ba$	2	$ma - ba = bd$
$2 \div m - b$	3	$a = \dfrac{bd}{m - b} = 12$, the Number of Days required, or the Time in which B will overtake A.

Queſtion

Question 110. *If a Scribe can in 8 Days write 15 Sheets, How many such Scribes can write 405 Sheets in 9 Days?*

Let $a =$ the Number of Scribes, $b = 8$, $d = 15$, $m = 405$, $n = 9$.

Then	1	$b : d :: n : \frac{dn}{b}$ the Number of Sheets the Scribe can write in nine Days.
and	2	$\frac{dn}{b} : 1 :: m : \frac{bm}{dn} =$ the Number of Scribes to write the 405 Sheets in nine Days.
hence	3	$a = \frac{bm}{dn} = \frac{3240}{135} = 24$, the Number (of Scribes required.

Question 111. A *can do a Piece of Work once in 3 Weeks,* B *can do it three times in 8 Weeks, and* C *can do it five times in 12 Weeks. In how long Time can they do it jointly?*

Let $a =$ the Time required, $b = 1$, $d = 3$, $g = 8$, $n = 5$, $m = 12$, the Number 3 occurring twice, I put only d for it.

Then	1	$d : b :: a : \frac{ba}{d}$, the Part of the Work that can be done by A in the Time sought.
and	2	$g : d :: a : \frac{da}{g}$, the Part of the Work that can be done by B in the Time sought.
and	3	$m : n :: a : \frac{na}{m}$, the Part of the Work that can be done by C in the Time sought.

And as these three Parts are to be equal to 1, or one Work,

therefore	4	$\frac{ba}{d} + \frac{da}{g} + \frac{na}{m} = 1.$
whence	5	$a = \dfrac{1}{\frac{b}{d} + \frac{d}{g} + \frac{n}{m}} = \dfrac{1}{\frac{1}{3} + \frac{3}{8} + \frac{5}{12}}$

by

by reducing the Fractions $\frac{1}{3}+\frac{3}{8}+\frac{5}{12}$ to a common Denominator, and adding and abbreviating them, we shall find $\frac{1}{3}+\frac{3}{8}+\frac{5}{12}=\frac{9}{8}$.

Whence $a=\frac{1}{\frac{9}{8}}=\frac{8}{9}$ of a Week, by the Rule for Division (of Vulgar Fractions.

If the Week consists of 6 Days

$$\begin{array}{l} \quad 8 \\ \hline 9)\ 48\ (5 \text{ Days} \\ \quad 45 \\ \hline \quad 3 \end{array}$$

And the Days consist of 12 Hours

$$\begin{array}{l} 9)\ 36\ (4 \text{ Hours,} \\ \quad 36 \end{array}$$

that is, they will perform the Work in five Days four Hours.

Or the Equation $\frac{ba}{g}+\frac{da}{g}+\frac{na}{m}=1$, may be reduced thus;

$4\times d$	6	$ba+\frac{dda}{g}+\frac{dna}{m}=d$
$6\times g$	7	$gba+dda+\frac{gdna}{m}=gd$
$7\times m$	8	$mgba+mdda+gdna=mgd$
$8\div$	9	$a=\frac{mgd}{mgb+mdd+gdn}=\frac{288}{324}$
		$=\frac{8}{9}$ of a Week as above.

73. Having in this easy familiar Manner, by general and universal Rules, explained to the Learner the Elements of this celebrated Science, it may not be improper to raise his Curiosity, and animate him to exercise his Judgment in the Choice of Quantities for the Solution of the same Question, to give an Instance how much the Solution of Questions becomes more neat and elegant, by a judicious Choice of representing the unknown Quantities. The Question and its Solution is from the ingenious Mr. JOHN WARD's *Young Mathematician's Guide*.

Question

Question 112. *A Man playing at Hazard, or Dice, won the first Throw just so much Money as he had in his Pocket; the second Throw he won the square Root of what he then had, and five Shillings more; the third Throw he won the Square of all he then had; after which his whole Sum was* 112 l. 16 s. 0 d. *What Money had he when he began to play?*

Suppose	1	$a =$ his first Sum.
then	2	$2a =$ his Sum after the first Throw.
and	3	$\sqrt{2a} : + 5 =$ his Winnings at the second Throw.
$2 + 3$	4	$\sqrt{2a} : + 2a + 5 =$ his Sum after the second Throw.
$4 \odot 2$	5	$2a + 4a\sqrt{2a} + 10\sqrt{2a} + 4aa + 20a + 25 =$ his Winnings at the third Throw.
$4 + 5$	6	$24a + 4a\sqrt{2a} + 11\sqrt{2a} + 4aa + 30 = 2256$ Shillings.

Now to avoid these *surd Quantities*, let us make a second Supposition; thus,

Let	1	$2aa =$ his first Sum.
then	2	$4aa =$ his Sum after the first Throw.
and	3	$2a + 5 =$ his Winnings at the second Throw.
$2 + 3$	4	$4aa + 2a + 5 =$ his Sum after the second Throw.
$4 \odot 2$	5	$16aaaa + 16aaa + 40aa + 20a + 4aa + 25 =$ his Winnings at the third Throw.
$4 + 5$	6	$16aaaa + 16aaa + 48aa + 22a + 30 = 2256$

But to avoid these high Equations, let us make a third Supposition; thus,

Let	1	$\frac{aa}{2} =$ his first Sum.
then	2	$aa =$ his Sum after the first Throw.
and	3	$a + 5 =$ his Winnings at the second Throw.
$2 + 3$	4	$aa + a + 5 =$ his Sum after the second Throw.

But

But as it was the Square of $aa+a+5$ he won at the third Throw, to avoid the Trouble of squaring it,

substitute	5	$e=aa+a+5$
then	6	$ee=$ his Winnings at the third Throw, consequently,
$5+6$	7	$ee+e=2256$ Shillings
$7\ c\ \square$	8	$ee+e+0.25=2256.25$
$8\ \omega\ 2$	9	$e+0.5=47.5$
$9-0.5$	10	$e=47$. Because at the fifth Step e was substituted for $aa+a+5$
$5 \cdot 10$	11	$aa+a+5=47$
$11-5$	12	$aa+a=42$
$12\ c\ \square$	13	$aa+a+0.25=42.25$
$13\ \omega\ 2$	14	$a+0.5=6.5$
$14-0.5$	15	$a=6$
whence	16	$\frac{aa}{2}=18$ Shillings, the Money he had (when he first began to play.

The Learner will easily observe, that the third Solution is more neat and elegant than either of the other two; tho' I know of no *general* Rule that is given for the Choice of the Quantities to state the Question, but it is left to the Judgment and Sagacity of the Reader, and as such Methods must be attended with particular Difficulties to a Learner, I have avoided the perplexing him with them; but as he has now a general Method of solving Equations, he may exercise his Judgment at his own Discretion, in the Choice of different Quantities to represent the same Question.

The Method of expressing the Power of any Quantity, by placing a Figure over it.

74. THERE is a more compendious Method of expressing the high Powers of any Quantity, than writing them at length, by placing a Figure over the Quantity thus, $\overset{4}{a}$ is $aaaa$, and $\overset{3}{a}$ is aaa, and $\overset{1}{a}$ is a, and $\overset{2}{a}\overset{3}{b}$ is $aabbb$, that is, the Figure that stands over the Letter shows to what

Power that Letter, or Quantity, is involved, which Method of Notation is generally used when the Powers are high. The Figures placed over the Quantity are called *Exponents*. The Mind being a little accustomed to this Method of Notation, will as easily manage an *Algebraic* Process, when the Powers are expressed by *Exponents*, as if they were repeated at length; and for the further Ease of the Learner, in this Method of Notation, we will resume the Solution of Question 90, expressing the Powers by *Exponents*, that the Learner may compare both the Operations together.

	1	$a^2 + a - e^2 = m$ } to find a and e.
	2	$e^2 + e = 2a$
$1 + e^2$	3	$a^2 + a = m + e^2$
$3 - m$	4	$a^2 + a - m = e^2$
$2 - e$	5	$e^2 = 2a - e$
$4 \cdot 5$	6	$a^2 + a - m = 2a - e$
$6 + e$	7	$a^2 + a - m + e = 2a$
$7 - a$	8	$a^2 - m + e = a$
$8 + m$	9	$a^2 + e = a + m$
$9 - a^2$	10	$e = a + m - a^2$
$10 \circledast 2$	11	$e^2 = a^2 + 2am + m^2 - 2a^3 - 2ma^2 + a^4$
$2 \cdot 11 \cdot 10$	12	$a^2 + 2am + m^2 - 2a^3 - 2ma^2 + a^4 + a + m - a^2 = 2a$
12 in Numbers	13	$188a + 8836 - 2a^3 - 188a^2 + a^4 + a + 94 = 2a$
13 contracted	14	$187a + 8930 - 2a^3 - 188a^2 + a^4 = 0$
$13 - a^4$	15	$-a^4 = 187a + 8930 - 2a^3 - 188a^2$
$15 + 2a^3$	16	$-a^4 + 2a^3 = 187a + 8930 - 188a^2$
$16 + 188a^2$	17	$-a^4 + 2a^3 + 188a^2 = 187a + 8930$
$17 - 187a$	18	$-a^4 + 2a^3 + 188a^2 - 187a = 8930$

In the same Manner the Learner may attempt the Solution of any of the other Questions, expressing the Powers by *Exponents*: One Thing is to be carefully observed, that the *Exponent* belongs only to the Letter which stands under it, and when it is only *Unity*, or 1, it is never set down, like the Co-efficient when it is *Unity* only, it is generally omitted in the Expression.

The Method of knowing if a Queſtion is limited, or admits but of one Anſwer; or if it is indetermined, that is, admits of ſeveral Anſwers.

75. THE Queſtion being ſtated, that is, all the Equations being expreſſed which are neceſſary for the Solution of the Queſtion, then if there are *more* unknown Quantities than there are *Equations*, the Queſtion admits of a *Variety* of Anſwers, and is therefore unlimited or indetermined, *ex. gr.*

$$\left.\begin{array}{r}\text{Suppoſe } a+e=40\\ \text{And } e+y=20\end{array}\right\}\text{ to find } a,\ e,\text{ and } y.$$

Here there are three unknown Quantities, and only two Equations.

Now e being in both the given Equations, you may ſuppoſe it any Number under 20, the leaſt of the two given Numbers, as for Example ſuppoſe $e=16$.

Then the firſt Equation is $a+16=40$.
And the ſecond Equation is $16+y=20$.

From whence it will be eaſy to find a and y, but if e is ſuppoſed any other Number under 20, then there will be found different Numbers for a and y, and the like of any other Queſtion, where the Number of unknown Quantities, are more than the Equations which ariſe from the Queſtion.

But when the Number of given Equations are *juſt as many as the unknown Quantities required to be found*, then the Queſtion generally admits but of *one* Anſwer, for then each of the Quantities ſought hath generally but *one* ſingle Value, thus as at Queſtion 80, where we have

$$\begin{array}{|c|l}1 & a+e+y=b=18\\ 2 & a+3e-2y=m=9\\ 3 & a+4y-2e=p=21\end{array}$$

Where $a=5$, $e=6$, and $y=7$.

But

But when the Number of given Equations *exceeds the Number of Quantities sought*, they not only limit the Question, but often render it *impossible*, as one of the Equations may be inconsistent with another; as for Example,

$$\left.\begin{array}{l|l} 1 & a+e=16 \\ 2 & ae=48 \\ 3 & a-e=22 \end{array}\right\} \text{to find } a \text{ and } e.$$

Now here are three Equations, and but two unknown Quantities, and the first and second Equations include a possible Case, and it may be found what the Numbers are.

And if we take the second and third Equations, they likewise include a possible Case, for it may be determined what those Numbers are.

But all three Equations together render the Case impossible, the first Equation being incompatible with the third, as the Sum of two Numbers cannot be less than their Difference.

To raise or invent a Method to extract the Cube Root.

76. THIS is no more than the Method of *Converging Series* applied to the Solution of an Equation, one Side of which is the unknown Quantity, and is a pure Cube, or raised to the third Power only, *ex. gr.*

Suppose $aaa = 9261$, where 9261 is a Cube Number, now to find what a, or the Number is that being cubed will produce 9261, is to extract the Cube Root of 9261.

By the common Method of distinguishing of how many Places the Root will consist, by placing a Point over the Place of Units, and another over every third Figure, the Root will consist of two Places, therefore suppose the Cube Root to be - - - - 20

$$\begin{array}{r} 20 \\ 20 \\ \hline 400 \\ 20 \\ \hline 8000 \end{array}$$

which being less than 9261 the given Number, the Cube Root of 9261 must be more than 20.

Now put $r = 20$, and e for what 20 wants of the true Root, then is $r + e = a$, or the Cube Root of 9261, and proceed as in the Method of *Converging Series*, Case 1. Page 230.

If	1	$r + e = a$,

Raise this Equation to the third Power, because it is the Cube Root, which is to be extracted.

1 ⓖ 3	2	$rrr + 3rre + 3ree + eee = aaa$
but	3	$aaa = 9261$ by the Example,
2 . 3	4	$rrr + 3rre + 3ree + eee = 9261$

Put this Equation into Numbers, and reject all the Powers of e above ee, as in the Method of *Converging Series*.

4 in Numbers	5	$8000 + 1200e + 60ee = 9261$
		Because 8000 is less than 9261, transpose 8000
5 — 8000	6	$1200e + 60ee = 1261$
		Dividing by the Co-efficient of ee.
6 ÷ 60	7	$20e + ee = 21.01$
		Dividing by $20 + e$, that is, by the Co-efficient of e *plus* e, as in the Method of *Converging Series*,
7 ÷ 20 + e	8	$e = \frac{21.01}{20+e}$

Operation in Numbers,

$$\begin{array}{rrl} & 20) & 21.01\ (1 = e \\ +e = & 1 & \\ \text{Divisor } & 21 & 21 \\ & & .01 \text{ Remainder rejected.} \end{array}$$

$$\begin{array}{r} r = 20 \\ +e = \ \ 1 \\ \hline \end{array}$$

$r + e = 21 = a$, which being tried will be found to be the Cube Root of 9261. And by the same Method may the Cube Root of any other Number be extracted.

But to save the Trouble of repeating this Operation, when any Cube Root is to be extracted, the above Process may be made more general, by not turning the Equation at the fourth Step into Numbers, and putting any Letter for the given Number, whose Cube Root is to be extracted.

Suppose

Suppose as before $aaa = 9261$, let $b = 9261$.

Then $aaa = b$, to find a, or to extract the Cube Root.

Now make a Supposition that 20 is the Root, which being tried as before, it will be found too little. Then put $20 = r$, and because 20 is too little, r in this Case is usually called *less than just*; and for what r wants of the true Root put e, whence $r + e$ will be the true Root, or equal to a.

Hence	1	$r + e = a$ Raise this Equation to the third Power as before.
1 & 3	2	$rrr + 3rre + 3ree + eee = aaa$
but	3	$aaa = b$ as above
2 . 3	4	$rrr + 3rre + 3ree + eee = b$

As we know rrr to be less than b, by finding the Cube of 20 was less than the given Number, therefore transpose rrr, and reject the Powers of e above ee.

$4 - rrr$	5	$3rre + 3ree = b - rrr$ Dividing by the Co-efficient of ee,
$5 \div 3r$	6	$re + ee = \dfrac{b - rrr}{3r}$

As there will be another Division before the Operation is finished, to keep the Fraction as simple as may be, substitute $D = \dfrac{b - rrr}{3r}$.

Then	7	$re + ee = D$

Now dividing by $r + e$, that is, the Co-efficient of e *plus* e,

$7 \div r + e$	8	$e = \dfrac{D}{r + e}$, *THEOREM* 1.

Operation in Numbers,

$$\begin{array}{r} b = 9261 \\ -rrr = -8000 \\ \hline 3r = 60)\ 1261\ (21.01 = D. \\ 120 \\ \hline 61 \\ 60 \\ \hline 100 \\ 60 \\ \hline 40 \end{array}$$

$r = 20$.

$$
\begin{array}{rl}
r = 20) & 21.01 = D\ (1 = e. \\
+e = \quad 1 & \\
\text{Divisor } 21 & 21 \\
 & \overline{.01} \text{ Remainder neglected.}
\end{array}
$$

$$
\begin{array}{rl}
r = & 20 \\
+e = & 1 \\
\hline
r + e = & 21 = a, \text{ the Cube Root required as before.}
\end{array}
$$

Now suppose it was required to extract the Cube Root of 132651.

Here according to the Method of pointing, the Root will consist of two Places, and to make a tolerable near Supposition at the first Trial, the first Period being 132, I consider what whole Number cubed will be the nearest to 132, and I find it to be 5, then as the Root consists of two Places, I supply the next Place with a *Cypher*, and suppose the Root to be 50, which I know is less than the true Root, as the Cube of 5 is less than 132.

Hence, as before, we are to determine what the Number is, that 50 wants of the true Root of 132651.

Then putting $r = 50$, and e what it wants of the true Root, and $b = 132651$, we have just the same substituted Letters as in the last Example; and if the Operation was repeated it will be exactly the same, it is therefore needless to repeat the Work, but only observing the Equation, or *Theorem* to find e, which is $e = \frac{D}{r+e}$, and by Substitution we have $D = \frac{b - rrr}{3r}$.

$$
\begin{array}{rl}
\text{Now } b = & 132651 \\
-rrr = & -125000 \\
\hline
3r = 150) & 7651\ (51.006 = D. \\
 & \underline{750} \\
 & 151 \\
 & \underline{150} \\
 & 1000 \\
 & \underline{900} \\
 & 100
\end{array}
$$

$r = 50$

$r = 50)\ 51.006 = D\ (1 = e$
$+e = \underline{1}$
Divisor $51\ \ 51$
.006 Remainder neglected.

Now it was $r = 50$
We have found $e = \underline{\ \ 1}$
$r + e = 51$ the Cube Root of 132651, which being tried will be found to be true.

And in the same Manner, the Cube Root of any other Number may be extracted, without repeating the *Algebraic* Work, when the Number assumed for the Root is less than the *true* Root: But when the Number assumed for the Root is too much, or more than the true Root, then we proceed as in the following Example, in the same Manner as at the second *Case* of *Converging Series*, Page 235.

Required to extract the Cube Root of 24389, or $aaa = 24389$.

By the usual Method of pointing, the Root will consist of two Figures, the first Period of the given Number is 24, and the Cube of 3 being the nearest of whole Numbers to 24, and supplying the other Place of the Root with a *Cypher*, I suppose 30 to be the Cube Root of 24389, but the Cube of 30 is 27000, which being more than the given Number, the Cube Root cannot be so much as 30.

Therefore let $r = 30$, which is now too great or *more than just*, and what 30 is too much call e, then will $r - e = a$, or the true Cube Root required, and calling the given Number $24389 = b$,

we have	1	$r - e = a$ Raise this Equation to the third Power as before.
1 ⓖ 3	2	$rrr - 3rre + 3ree - eee = aaa$
but	3	$aaa = 24389 = b$, as b is put for the given Number.
2 . 3	4	$rrr - 3rre + 3ree - eee = b$

Because b is less than rrr, transpose b and reject the Powers of e above ee.

$4 - b$	5	$rrr - b - 3rre + 3ree = 0$, for one Side of the Equation substracted from the other Side must leave 0, or *nothing*.

 Then

Then tranſpoſe all the Powers of e, to the other Side of the Equation.

$5 + 3rre$	6	$3rre = rrr - b + 3ree$
$6 - 3ree$	7	$3rre - 3ree = rrr - b$
		Dividing by the Co-efficient of ee,
$7 \div 3r$	8	$re - ee = \frac{rrr - b}{3r}$

As there will be another Diviſion before the Operation is finiſhed, to keep the Fraction as ſimple as may be, ſubſtitute $G = \frac{rrr - b}{3r}$.

Then | 9 | $re - ee = G$

Now dividing by $r - e$, that is, by the Co-efficient of e *minus* e,

$9 \div r - e$ | 10 | $e = \frac{G}{r - e}$, *THEOREM* 2.

Operation,

$$\begin{array}{r}
rrr = 27000 \\
-b = -24389 \\
\hline
3r = 90)\ 2611\ (29.01 = G. \\
180 \\
\hline
811 \\
810 \\
\hline
100 \\
90 \\
\hline
10
\end{array}$$

$$\begin{array}{l}
r = 30)\ 29.01 = G\ (1 = e \\
\quad -1 \\
\text{Diviſor } 29 \quad 29 \\
\qquad\qquad .01 \text{ Remainder neglected.}
\end{array}$$

$$\begin{array}{rcl}
r &=& 30 \\
-e &=& -1 \\
\hline
r - e &=& 29 = a,
\end{array}$$

which being cubed will be found the true Cube Root of 24389.

In

In this Case, the Quotient Figure is substracted from the Divisor as it is found, the Divisor at the tenth Step being $r - e$, whereas in *Theorem* 1, p. 309, it was at the eighth Step $r + e$.

Now as the first supposed Root must be too great or too little, unless it happens to be taken exact at the first Time, therefore these two *Theorems* will extract the Cube Root of any Number, as in the following Example.

Let it be required to extract the Cube Root of 14526.784

From pointing the whole Numbers according to the usual Method in common Arithmetic, the Root will consist of two Places of Integers, the first Period of the given Number being 14, the Cube of the whole Number which is nearest to 14 is 2; and supplying the other Place of the Root with a *Cypher*, I suppose the Root of the given Number to be 20, which is too little, or *less than just*, the Cube of 2 the first Figure in the Root being less than 14, the first Period in the given Number.

Then putting $b = 14526.784$ $r = 20$, and e what 20 wants of the true Root, we proceed as at *Theorem* 1, page 309, where $e = \dfrac{D}{r+e}$, and by Substitution $D = \dfrac{b - rrr}{3r}$.

$$\begin{array}{rl}
b = & 14526.784 \\
-rrr = & -\ 8000. \\
\hline
3r = 60) & 6526.784\ (108.78 \text{ nearest} = D. \\
& 60 \\
\hline
& 526 \\
& 480 \\
\hline
& \ \ 467 \\
& \ \ 420 \\
\hline
& \ \ \ 478
\end{array}$$

$$\begin{array}{rrl}
r = 20) & 108.78 = D & (4.44 = e. \\
+e = 4 & & \\
\hline
\text{Divisor } 24 & 96 & \\
\hline
4.4 & 1278 & \\
\hline
\text{Divisor } 28.4 & 1136 & \\
\hline
.44 & 14200 & \\
\hline
\text{Divisor } 28.84 & 11536 & \\
\hline
& 2664 &
\end{array}$$

The Reader will obferve that the Quotient Figure is added twice to the Divifor to compleat it, in the fame Manner as at the Method of *Converging Series*, Page 233.

$$\begin{array}{rl} \text{Now } r = & 20 \\ +e = & 4.44 \\ \hline r+e = & 24.44 \end{array}$$

and to try whether this is the true Root of the given Number cube it.

$$\begin{array}{r} 24.44 \\ 24.44 \\ \hline 9776 \\ 9776 \\ 9776 \\ 4888 \\ \hline 597.3136 \\ 24.44 \\ \hline 23892544 \\ 23892544 \\ 23892544 \\ 11946272 \\ \hline 14598.344384 \end{array}$$

which being greater than the given Number, the Root cannot be fo much as 24.44

To approach ftill nearer to the true Root, make a fecond Operation, fuppofing the Number laft found, *viz.* 24.44 to be r, and put e for what that Number is too much, then $r - e$ will be the true Root, and putting the given Number 14526.784 $= b$, we proceed as at *Theorem* 2, Page 312, where $e = \frac{G}{r-e}$, and by Subftitution $G = \frac{rrr - b}{3r}$.

$$\begin{array}{rl} rrr = & 14598.344384 \\ -b = - & 14526.784 \\ \hline 3r = 73.32) & 71.560384 \; (.976 = G. \\ & 65988 \\ \hline & 55723 \\ & 51324 \\ \hline & 43998 \\ & 43992 \\ \hline & 6 \end{array}$$

$r = 24.$

$$r = 24.44)\ .9760 = G\ (.04 = e.$$
$$-e = -\ \ .04$$
Divisor 24.40   9760

0

Now $r = 24.44$ by the first Operation.

$-e = -\ .04$

$r - e = 24.4$ which being cubed, will be found the true Root of 14526.784

Therefore by the second Operation the true Root is found.

For a further Variety, let it be again required to extract the Cube Root of the same Number 14526.784

But let us suppose the Cube Root to be 30, the Cube of which being 27000, the Root cannot be so much as 30, then putting $r = 30$, we shall have r too great or *more than just*, and putting e what it is too much, then $r - e$ will be the true Root, and calling the given Number $14526.784 = b$, we proceed as at *Theorem* 2, Page 312, where $e = \frac{G}{r - e}$, and by Substitution $G = \frac{rrr - b}{3r}$.

$$rrr = 27000.$$
$$-b = -14526.784$$
$$3r = 90)\ 12473.216\ (138.591 = G.$$

```
 90
 347
 270
  773
  720
   532
   450
    821
    810
     116
      90
      26
```

$$\begin{array}{rl}
r = 30) & 138.591 = G\ (5.7 \\
-e = -5 & \\
\hline
\text{Divisor } 25 & 125 \\
 & \overline{1359} \\
-5.7 & \\
\hline
\text{Divisor } 19.3 & 1351 \\
 & \overline{\quad 8}
\end{array}$$

$$\begin{array}{rl}
r = & 30. \\
-e = & -5.7 \\
\hline
\end{array}$$

$r - e = 24.3$ to try whether this is the Cube Root of 14526.784 cube 24.3

$$\begin{array}{r}
24.3 \\
24.3 \\
\hline
729 \\
972\ \ \\
486\ \ \ \ \\
\hline
590.49 \\
24.3 \\
\hline
177147 \\
236196\ \ \\
118098\ \ \ \ \\
\hline
14348.907
\end{array}$$

which being less than the given Number 14526.784 the Cube Root must be more than 24.3

Now for a second Operation, and let $r = 24.3$ and what it wants of the true Root call e, then will $r + e$ be the true Root, and still calling the given Nnmber $14526.784 = b$, we now proceed as at *Theorem* 1, Page 309, where $e = \dfrac{D}{r + e}$, and by Substitution $D = \dfrac{b - rrr}{3r}$.

$b =$

$$\begin{array}{rl} b = & 14526.784 \\ -rrr = & -14348.907 \\ \hline 3r = 72.9) & 177.877\ (2.44 = D. \\ & 1458 \\ \hline & 3207 \\ & 2916 \\ \hline & 2917 \\ & 2916 \\ \hline & 1 \end{array}$$

$$\begin{array}{ll} r = 24.3) & 2.44 = D\ (.1 = e. \\ +\ \ .1 & \\ \hline \text{Divisor } 24.4 & \underline{244} \\ & 0 \end{array}$$

$$\begin{array}{rl} r = & 24.3 \text{ by the first Operation.} \\ +e = & .1 \\ \hline r + e = & 24.4 \text{ the true Root as before.} \end{array}$$

In the ſame Manner may the Cube Root of any other Number be extracted, and tho' the true Root may not always be exactly had, yet by repeating the Operation you may approach to it, within any aſſignable Degree of Exactneſs, and if a ſmall Miſtake happens in the firſt, it will be corrected at the ſecond Operation.

To extract the Biquadrate, or fourth Root.

THIS Operation proceeds in the ſame Manner as in the Cube Root, only raiſing the $r+e$ or $r-e$ to the fourth Power, thus,

Required the *Biquadrate* or fourth Root of 194481, or of $aaaa = 194481$.

By placing a Point over the Place of *Units*, and another over every fourth Figure, we ſhall find the Root will conſiſt of two Figures: And the firſt Period of the given Number being 19, now the Biquadrate or fourth Power of 2 being 16, which being the neareſt in Integers, and ſupplying the other Place of the Root with a *Cypher*, ſuppoſe 20 to be the Biquadrate Root of 194481; but the Biquadrate of 20 being only 160000, the Root muſt be more than 20. Now let $r = 20$, and putting e for what 20 wants of the true Root, then will $r+e = a$ be the true Root required; and calling the given Number $194481 = b$, then $aaaa = b$.

Now

Now	1	$r + e = a.$

Raiſe this Equation to the fourth Power, becauſe it is the *Biquadrate* Root that is to be extracted.

1 ☉ 4	2	$rrrr + 4rrre + 6rree = aaaa$, all the Powers of e above ee being rejected.
But	3	$aaaa = b$, b being put equal to the given Number.
2 . 3	4	$rrrr + 4rrre + 6rree = b.$ Becauſe $rrrr$ is leſs than b, tranſpoſe $rrrr$.
$4 - rrrr$	5	$4rrre + 6rree = b - rrrr$ Dividing by the Co-efficient of ee,
$5 \div 6rr$	6	$\frac{2re}{3} + ee = \frac{b - rrrr}{6rr}$

As there will be another Diviſion before the Operation is finiſhed, therefore as in the Cube Root, put $D = \frac{b - rrrr}{6rr}$.

Then	7	$\frac{2re}{3} + ee = D.$

Now dividing by $\frac{2r}{3} + e$, that is, the Co-efficient of e *plus* e,

$8 \div \frac{2r}{3} + e$	8	$e = \dfrac{D}{\frac{2r}{3} + e}$ *THEOREM* I.

Operation $b = 194481$
$- rrrr = - 160000$
$6rr = 2400)\ 34481\ (14.367 = D.$
2400
10481
9600
8810
7200
16100
14400
17000

2r

$$\frac{2r}{3} = 13.33) \quad 14.367 = D \ (1. = e.$$

$$+ \ 1.$$

$$14.33 \quad 1433$$

$$37 \text{ Remainder neglected.}$$

$$r = 20$$

$$+e = \ 1$$

$r + e = 21 = a$, which being raised to the fourth Power, will be found to be the *Biquadrate* Root of the given Number.

And if we here take the first Root too great, or more *than the Truth*, the Operation is the same as raising the second *Theorem* for the Cube Root.

Suppose $aaaa = 456976$, to find the *Biquadrate* Root.

The Root being found to consist of two Figures as before, and the first Period in the given Number being 45, and the *Biquadrate* of 3 being 81, supplying the other Place of the Root with a *Cypher*, let us suppose 30 to be the Root, but the *Biquadrate* of 30 is 810000, which being more than 456976, the Root cannot be so much as 30.

Then putting $r = 30$, and e what 30 is too much, we have $r - e = a$ the Root required; and putting $b = 456976$, we then have $aaaa = b$.

Now	1	$r - e = a$ Raising this Equation to the fourth Power as before, and neglecting the Powers of e above ee.
1 ☉ 4	2	$rrrr - 4rrre + 6rree = aaaa$
But	3	$aaaa = b$
2 . 3	4	$rrrr - 4rrre + 6rree = b$

Because b is less than $rrrr$ therefore transpose b.

$4 - b$	5	$rrrr - b - 4rrre + 6rree = 0$, one Side of the Equation being substracted from the other, must leave *nothing*, now transpose the several Powers of e.
$5 + 4rrre$	6	$4rrre = rrrr - b + 6rree$

6 —

$6-6rree$	7	$4rrre-6rree=rrrr-b$ Divide by the Co-efficient of ee.
$7\div 6rr$	8	$\frac{2re}{3}-ee=\frac{rrrr-b}{6rr}$

For the same Reason as in the last Operation, substitute $G=\frac{rrrr-b}{6rr}$.

Then	9	$\frac{2re}{3}-ee=G$ Now divide by $\frac{2r}{3}-e$, that is, by the Co-efficient of e *less* e.
$9\div\frac{2r}{3}-e$	10	$e=\frac{G}{\frac{2r}{3}-e}$ THEOREM 2.

Operation,

$$
\begin{array}{rl}
rrrr= & 810000 \\
-b=- & 456976 \\
6rr=5400) & 353024\ (65.374=G. \\
 & 32400 \\
 & 29024 \\
 & 27000 \\
 & 20240 \\
 & 16200 \\
 & 40400 \\
 & 37800 \\
 & 26000 \\
 & 21600 \\
 & 4400
\end{array}
$$

$$
\begin{array}{rl}
\frac{2r}{3}=20.) & 65.374=G\ (4.08=e. \\
-4 & \\
\text{Divisor } 16. & 64 \\
-.08 & 13740 \\
\text{Divisor } 15.92 & 12736 \\
 & 1004
\end{array}
$$

$r=30$

$$\begin{array}{r} r = 30 \\ -e = -4.08 \\ \hline \end{array}$$

$r - e = 25.92 = a$, and to try if this is the Root, raise 25.92 to the fourth Power.

$$\begin{array}{r}
25.92 \\
25.92 \\ \hline
5184 \\
23328 \\
12960 \\
5184 \\ \hline
671.8464 \\
671.8464 \\ \hline
26873856 \\
40310784 \\
26873856 \\
53747712 \\
6718464 \\
47029248 \\
40310784 \\ \hline
451377.58519296
\end{array}$$

451377.58519296 which being less than the given Number 456976, the true Root must be more than 25.92

Then for a second Operation let $r = 25.92$ and for what it wants of the true Root put e, that now $r + e = a$, and still calling the given Number $456976 = b$, this is exactly the same Case as when we raised the first *Theorem*, Page 318, for the *Biquadrate* Root, whence we have no Occasion to repeat the *Algebraic* Work, but to use that *Theorem*, where $e = \dfrac{D}{\dfrac{2r}{3} + e}$, and by Substitution $D = \dfrac{b - rrrr}{6rr}$.

 $b =$

$$b = 456976.$$
$$-rrrr = -451377.5852$$

$$6rr = 4031.0784)\ 5598.4148\ \ (1.3888 = D.$$
$$\begin{array}{r} 40310784 \\ \hline 156733640 \\ 120932352 \\ \hline 358012880 \\ 322486272 \\ \hline 355266080 \\ 322486272 \\ \hline 32779808 \end{array}$$

$$\frac{2r}{3} = 17.28)\ 1.3888 = D\ (.08 = e.$$
$$+\ \ .08$$
$$\text{Divisor } 1736\quad 13888$$
$$0$$

$$r = 25.92 \text{ by the first Operation.}$$
$$+e = \ \ .08$$

$r + e = 26. = a$, which being involved to the fourth Power, will be found the true *Biquadrate* Root of 456976.

The Reader will easily observe that these two *Theorems* will extract the *Biquadrate* Root of any given Number, in the same Manner as the two *Theorems* did for the Cube Root.

In the same Method may *Theorems* be raised to extract any Root, it being no more than to suppose a Number to be the required Root, and try whether it is too great or too little; then calling it $r + e$, or $r - e = a$, or the true Root, as the Occasion requires, and raise this Equation as high as the Root is to be extracted, after which the Operation is the same as before.

To turn Equations into Analogies.

77. SUPPOSE there was given this Proportion $a : b :: c : d$; then multiplying Extreams and Means we have this Equation $ad = bc$, now as we get an Equation from Quantities in continual Proportion, by multiplying the Extreams and Means, and making one Product equal to the other. Hence to turn any Equation into an Analogy, is only the reverfe, by taking the Quantities that compofe either Side of the Equation, and making them the two Extreams, and the Quantities that compofe the other Side of the Equation, and making them the two Means in the Proportion.

To turn the Equation $md = za$ into an Analogy.

One Side of the Equation is compofed of the Quantities m and d.

And the other Side of the Equation is compofed of the Quantities z and a.

Hence placing thefe Quantities according to the Direction, we have $m : z :: a : d$

Or $d : z :: a : m$
Or $z : d :: m : a$
Or $z : m :: d : a$, &c.

For multiplying the Extreams and Means of either of thefe Proportions, we fhall ftill have the given Equation $dm = za$.

Again, fuppofe the Equation $an = bdx$, and it is required to find the Proportion of a to b.

Now one Side of the Equation is compofed of the Quantities a and n.

And the other Side of the Equation is compofed of the Quantities b and dx.

But in ranging thefe Quantities, make the Quantities a and whofe Proportion is required, the firft and fecond Terms in the Proportion, and place the other two Quantities fo, that if the Extreams and Means were to be multiplied, they will produce the given Equation, and then we fhall find $a : b :: dx : n$.

From the Equation $dny = bxz$, to find the Proportion of d to b.

By the Directions we fhall find $d : b :: xz : ny$.

From $\frac{an}{m} = pxd$, to find the Proportion of a to d, which is $a : d :: px : \frac{n}{m}$ for multiplying Extreams and Means $\frac{an}{m} = dpx$.

To find the Proportion of a to z from $\frac{an}{y} = \frac{bz}{x}$, here $a : z :: \frac{b}{x} : \frac{n}{y}$.

To find the Proportion of ab to d, from ab or $1\,ab = dny$.

Here $ab : d :: ny : 1$, for multiplying Extreams and Means we have $ab = dny$.

78. I ſhall now ſhow the Learner, the Certainty of the Rules on which this Science is founded; this I have purpoſely omitted in the Beginning of the Work, imagining it unreaſonable to expect a Learner to ſee the Force of a Demonſtration in *Algebra*, before he is acquainted with its Characters and Language.

The Foundation of tranſpoſing Quantities.

THIS is grounded on the obvious Truth, *that every Thing is equal to itſelf*; that is, $m = m$, and $-y = -y$; whence to tranſpoſe any Quantity, is only to make that Quantity equal to itſelf, prefixing to it the contrary Sign, and adding it to the given Equation. Suppoſe there is given

	1	$a - b + d - m = z$, to tranſpoſe, b, d, and m.
Now	2	$b = b$
1 + 2	3	$a + d - m = z + b$
And	4	$-d = -d$
3 + 4	5	$a - m = z + b - d$
Laſtly	6	$m = m$
5 + 6	7	$a = z + b - d + m$

SUBSTRACTION.

I ſay to ſubſtract a *negative* Quantity from a *poſitive*, is only to change the Sign of the *negative* Quantity, and *add* it to the

poſitive

positive Quantity, and this Sum will be the *Remainder* required. That is,

If $x-y$ is subtracted from $x+y$, I say the Remainder is $2y$, for

Suppose	1	$x+y=m$
And	2	$x-y=n$

Now if it can be proved that $2y$ is equal to the Difference between m and n, it follows that to subtract a *negative* Quantity is to change its Sign and add it.

$2+y$	3	$x=n+y$

Now in the first Equation for x write $n+y$, for that is equal to x.

Then	4	$n+2y=m$
$4-n$	5	$2y=m-n$. Q. E. D.

I say further, that to subtract a *negative* Quantity from a *negative* Quantity, is done by *changing* the Sign of the Quantity to be subtracted, and then *adding* them by the Rules in Addition, and the Sum will be the *Difference* required.

Suppose	1	$x-2y=m$
And	2	$x-y=n$

Now the second Equation being subtracted from the first according to the Rule, leaves $-y=m-n$; and if it can be proved that $-y=m-n$, then to subtract a *negative* Quantity from a *negative* Quantity, is only to change its Sign and add it.

$2+y$	3	$x=n+y$
$1 \cdot 3$	4	$n+y-2y=m$
$5-n$	5	$y-2y=m-n$
That is	6	$-y=m-n$. Q. E. D.

And that $m-n$ is a *negative* Quantity is evident, for $x-2y$ cannot be so great as $x-y$, they being supposed positive Quantities, and therefore m cannot be so great as n; consequently

sequently $m-n$ is a *negative* Quantity, and therefore may be equal to $-y$. And in

MULTIPLICATION,

I say unlike Signs being multiplied give $-$ in the Product; that is, $-a \times a = -aa$.

To prove which, I take for granted the following

LEMMA.

That no Quantities connected by the Sign $+$ only, or by the Sign $-$ only, can be equal to *nothing*. That is, it cannot be $-a-b=0$, or $a+b=0$, though it may be $a-b=0$, or $b-a=0$.

Now, if possible, let $-a \times a$ produce aa where the Sign of the Product is *affirmative*.

Let	1	$m-a=0$
	2	$a=a$, that is, every Quantity is equal to itself.
1×2	3	$ma+aa=0a$, by the Supposition,

that is, $ma+aa$ is equal to *nothing*, which is against the *Lemma*, therefore $-a \times a$ cannot produce aa.

But, I say $-a \times a = -aa$.

Let	1	$m-a=0$
	2	$+a=+a$, for any Quantity is equal to itself.
1×2	3	$ma-aa=0a$, that is, $ma-aa$ is

equal to *nothing*, whence $ma=aa$. Now that $ma=aa$ is evident, for $m-a=0$, therefore $m=a$, and multiplying by a we have $ma=aa$, consequently $-a \times a = -aa$. Q. E. D.

I say further, that like Signs tho' $-$ being multiplied, produce $+$ in the Product. That is, $-a \times -a$ produces aa, and not $-aa$, for

Let	1	$m-a=0$
	2	$-a=-a$, every *negative* Quantity being equal to itself.

Now, if possible, let $-a \times -a$ produce $-aa$. Then

1×2	3	$-ma-aa=-0a$, by the Suppo-

sition, that is, $-ma-aa$ is equal to *nothing*, which is against the

the *Lemma*, therefore $-a \times -a$ cannot produce $-aa$; but the Sign must be + or *affirmative*; which may be further proved thus,

Let	1	$m - a = 0$
	2	$-a = -a$
1×2	3	$-ma + aa = 0a$, that is, $-ma + aa$

is equal to *nothing*, from whence $ma = aa$. And that $ma = aa$ is evident, for $m - a = 0$, therefore $m = a$, and multiplying by a, we have $ma = aa$. Hence $-a \times -a = aa$. Q. E. D.

DIVISION.

As unlike Signs in Multiplication produce — in the Product, I say that,

In Division, unlike Signs being divided, give — in the Quotient, that is, if $ab - bb = 0$, and both Sides of the Equation be divided by b, I say the Quotient will be $a - b$, and not $a + b$.

Suppose unlike Signs to give + in the Quotient.

If	1	$ab - bb = 0$
Let	2	$b = b$
$1 \div 2$	3	$a + b = \frac{0}{b}$, by the Supposition, that is

$a + b$ is equal to *nothing*, which is against the *Lemma*, therefore an Absurdity follows the Supposition, that unlike Signs give + in the Quotient; but I say unlike Signs give — in the Quotient.

Let	1	$ab - bb = 0$
	2	$b = b$
$1 \div 2$	3	$a - b = \frac{0}{b}$, that is, $a - b$ is equal to

nothing, whence $a = b$, and that $a = b$ is thus proved.

$1 + bb$	4	$ab = bb$
$4 \div b$	5	$a = b$. Q. E. D.

I say further, that like Signs being divided, though they are *negative*, give + in the Quotient, that is, $ab - bb$ divided by $-b$, the Quotient is $-a + b$, and not $-a - b$.

If

If like Signs though — give — in the Quotient; then

$$\begin{array}{r|c|l} \text{Let} & 1 & ab - bb = 0 \\ & 2 & -b = -b \\ 1 \div 2 & 3 & -a - b = \frac{0}{b}, \text{ by the Suppoſition, that} \end{array}$$

is, $-a-b$ is equal to *nothing*, which is againſt the *Lemma*, therefore an Abſurdity follows the Suppoſition, that like Signs though — give — in the Quotient.

But, I ſay, $ab - bb$ divided by $-b$, the Quotient is $-a+b$, that is, like Signs though — give + in the Quotient. For,

$$\begin{array}{r|c|l} \text{Let} & 1 & ab - bb = 0 \\ & 2 & -b = -b \\ 1 \div 2 & 3 & -a + b = \frac{0}{b}, \text{ that is, } -a+b \text{ is} \end{array}$$

equal to *nothing*, whence $b = a$, and that $b = a$ is evident, thus,

$$\begin{array}{r|c|l} 1 + bb & 4 & ab = bb \\ 4 \div b & 5 & a = b. \quad \text{Q. E. D.} \end{array}$$

By this the Learner will ſee that like Signs though — both in Multiplication and Diviſion, muſt give + in the Product and Quotient, for an Abſurdity follows the contrary Hypotheſis, or Suppoſition, of their producing — in either the Product or Quotient.

The other Principles of this Science are very obvious, being the plain Conſequences of the *Axioms* mentioned in the Beginning of the Work.

FINIS.